Focusing properties of discrete RF quadrupoles *

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Abstract: The particle motion equation for a Radio Frequency (RF) quadrupole is derived. The motion equation shows that the general transform matrix of a RF quadrupole with length less than or equal to $0.5\beta\lambda$ (β is the relativistic velocity of particles and λ is wavelength of radio frequency electromagnetic field) can describe the particle motion in an arbitrarily long RF quadrupole. By iterative integration, the general transform matrix of a discrete RF quadrupole is derived from the motion equation. The transform matrix is in form of a power series of focusing parameter B. It shows that for length less than $\beta\lambda$, the series up to the 2nd order of B agrees well with the direct integration results for B up to 30, while for length less than $0.5\beta\lambda$, the series up to 1st order is already a good approximation of the real solution for B less than 30. The formula of the transform matrix can be integrated into linac or beam line design code to deal with the focusing of discrete RF quadrupoles.

Keywords: RF quadrupole, transform matrix, focusing parameter, iterative integration, hybrid RFQ

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1 Introduction

Inspired by the idea of separate accelerating and focusing, which was first implemented in Interdigital H-type Drift Tube Linac (IH-DTL) structures [1], a new accelerating structure named the Hybrid RF Quadrupole (H-RFQ) was proposed by P.N. Ostroumov in 2005 [2]. An H-RFQ consists of an alternating series of drift tubes (DTLs) and radio frequency quadrupole (RFQ) sections, which are incorporated into one resonator. It has both the high acceleration efficiency of IH-DTL structures and the efficient focusing of low energy ion beams of a radio frequency (RF) quadrupole. Because of its superiority in accelerating low charge heavy ion beams, it has attracted attention from a number of labs [3] since its invention

One of the most important issues in designing the H-RFQ structure is to describe the focusing effect of the RF quadrupoles precisely. The focusing properties of RF quadrupoles have been studied intensively during the development of RFQ accelerators in the 1970s, and one of the most important conclusions is that the transverse focusing strength of a unit cell is constant for a uniform RF quadrupole structure. P.N. Ostroumov applied this to the discrete RF quadrupole and used a static magnetic quadrupole to replace the RF quadrupole [2]. He assumed that the RFQ sections consist of cells with length exactly equal to $\beta\lambda/2$, and the focusing effect of the RFQ cells is equivalent to a magnetic quadrupole of length $\beta\lambda/2$ if

$$R_{0i}=\sqrt{\frac{2U_1}{\pi\beta c G_{\rm m}}}\ , \tag{1}$$

where U_1 is the electric potential on the RFQ electrodes and G_m is the equivalent magnetic gradient. Clearly it takes the RFQ cell as a static electric quadrupole with potential equal to the average potential within a half RF period

$$V_{\rm e}q=\frac{\displaystyle\int_{-\beta\lambda/4}^{\beta\lambda/4}U_1\cos kz{\rm d}z}{\displaystyle\int_{-\beta\lambda/4}^{\beta\lambda/4}{\rm d}z}=(2U_1)/\pi. \tag{2}$$

The magnetic quadrupole with equal focusing strength has magnetic gradient

$$G_{\rm m}=\frac{2U_1}{\pi\beta c}\frac{1}{R_{0i}^2}. \tag{3}$$

If we investigate the assumptions made in Ref. [2] carefully, we will find that they do not stand on a solid foundation. Firstly, the length of the RFQ cell is not necessarily equal to $\beta\lambda/2$ if the accelerating gap length is taken into account. For IH structures, the gap length

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is normally about 1/3 to 1/2 of the period length, so the RFQ cell length is only 2/3 or 1/2 of $\beta\lambda/2$ if the synchronous phases of the gaps on the both sides of the RFQ section are equal. Furthermore, in order to avoid RF sparking, the gap lengths between the drift tube and RF quadrupole section are even bigger, which makes the length of the RFQ cell even shorter. Secondly, for a discrete RFQ cell, the focusing strength depends on the RF phase when the particle enters the RFQ section, so it cannot be equivalent to a static magnetic quadrupole. In this paper, the general solution of a RF electric quadrupole in matrix form will be derived, which can be integrated into the linac design code.

2 Motion equation and transform matrix

The cross section of a RF electric quadrupole is shown in Fig. 1. We define x and y axes as shown in Fig. 1, and the z axis is directed outward from the paper from origin of the xy axes. If we choose the time when particles enter the RFQ as the zero of time, since the electrodes are uniform in z and the electric field lines are perpendicular to the z direction, the particle velocity along the z direction is constant, so we have

$$\omega t = kz, \tag{4}$$

where ω is the angular frequency of the RF field, and k is the wave number and is equal to $2\pi/\beta\lambda$. The electric potential of electrodes in the xz plane is

$$V_{\mathrm{RFQ}} = \frac{V_0}{2}\cos(kz+\varphi_0), \tag{5}$$

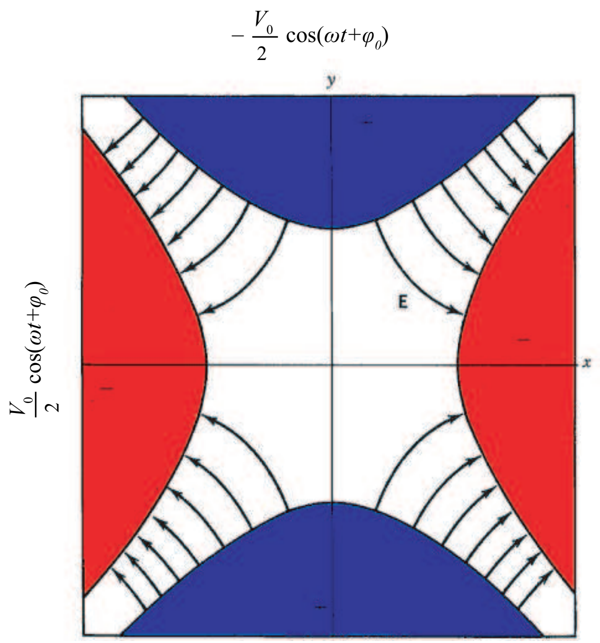

Fig. 1. Cross section of RF quadrupole.

where φ_0 is the initial RF phase when particles enter the RFQ. The potential of the electrodes in the yz plane is $-V_{\mathrm{RFQ}}$. Since the motions in the xz and yz planes are independent, we just consider the motion in the xz plane in the following. The motion equation can be derived from Newton's law. If the distance along the z axis is taken as an independent variable, then the motion equation in the xz plane is

$$\frac{\mathrm{d}^2x}{\mathrm{d}z^2} + \frac{1}{\gamma mc^2\beta^2}\frac{qV_0}{a^2}\cos(kz+\varphi_0)x = 0, \tag{6}$$

where γ is the relativistic parameter, q is the charge of the particle and a is the aperture radius of the quadrupole. For the yz plane, we just need to replace V_0 and x with $-V_0$ and y, respectively. This equation is the famous Mathieu equation [4] and has been studied intensively. It has four kinds of solutions and they are very complicated in form. In order to get a set of solutions which can be expressed in the form of a transfer matrix, so that it can be easily integrated in the linac design code, we need to rewrite Eq. (6) in another form. The well-known transverse focusing parameter B of a RFQ accelerator is defined as [5]

$$B = \frac{qXV_0}{\gamma mc^2}\frac{\lambda^2}{a^2}, \tag{7}$$

where X is the focusing parameter. In our case, there is no modulation of the electrodes, so $X=1$. The motion equation can be written as two first order equations

$$\frac{\mathrm{d}x}{\mathrm{d}z} = x', \tag{8}$$

$$\frac{\mathrm{d}x'}{\mathrm{d}z} = -\frac{B}{\beta^2\lambda^2}\cos(kz+\varphi_0)x. \tag{9}$$

Suppose the transform matrix of Eq. (9) is R. Then the coordinates at z are related to the initial coordinates by R as,

$$\begin{pmatrix} x(z) \\ x'(z) \end{pmatrix} = R(z,\varphi_0,B)\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}, \tag{10}$$

where R is a matrix function of z, initial phase φ_0 and focusing parameter B. Defining $R(0.5\beta\lambda,\varphi_0,B)$ as the transform matrix of a quadrupole with length $0.5\beta\lambda$ and initial phase φ_0, we have the following relation from the motion equation,

$$\begin{pmatrix} (x(z+0.5\beta\lambda,\varphi_0,B) \\ x'(z+0.5\beta\lambda,\varphi_0,B) \end{pmatrix} = R(z,\varphi_0+\pi,B)R(0.5\beta\lambda,\varphi_0,B)\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}. \tag{11}$$

This indicates that once we obtain the general transform matrix of a RF quadrupole with length less than or equal to $0.5\beta\lambda$, then with Eq. (11) we have the transform matrix of a RF quadrupole with arbitrary length.

Suppose the initial conditions of the particle are (x_0,x_0'). For the first step, we suppose that the velocity of particle is kept constant and is x_0', i.e.,

$$x'=x_0'=a_{21}^{(0)}x_0+a_{22}^{(0)}x_0'. \tag{12}$$

Taking Eq. (12) into Eq. (8), we have

$$x=x_0+x_0'z=a_{11}^{(0)}x_0+a_{12}^{(0)}x_0'. \tag{13}$$

We define the transfer matrix of zero order on B as

$$R^0=\begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} \\ a_{21}^{(0)} & a_{22}^{(0)} \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}. \tag{14}$$

This is just the same as the transfer matrix of a drift space with length z. Now we take Eq. (13) as the first approximation of the transverse location of the particle and take it into Eq. (9) and integrate it. We thus obtain x' in the first order approximation on B, that is

$$\begin{aligned} x' &= x_0'-\frac{B}{\beta^2\lambda^2}\int_0^z\cos(kz+\varphi_0)(x_0+x_0'z)\mathrm{d}z \\ &= [a_{21}^{(0)}+a_{21}^{(1)}]x_0+[a_{22}^{(0)}+a_{22}^{(1)}]x_0'. \end{aligned} \tag{15}$$

Taking Eq. (15) into Eq. (8), we obtain x in the first order of approximation on B:

$$\begin{aligned} x &= x_0+\int_0^z\left([a_{21}^{(0)}+a_{21}^{(1)}]x_0+[a_{22}^{(0)}+a_{22}^{(1)}]x_0'\right)\mathrm{d}z \\ &= [a_{11}^{(0)}+a_{11}^{(1)}]x_0+[a_{12}^{(0)}+a_{12}^{(1)}]x_0', \end{aligned} \tag{16}$$

where

$$a_{11}^{(1)} = \frac{B}{(2\pi)^2}[\cos\varphi_0-\cos(kz+\varphi_0)-kz\sin\varphi_0], \tag{17a}$$

$$a_{12}^{(1)} = \frac{B}{(2\pi)^2}\left[z\cos\varphi_0+z\cos(kz+\varphi_0)+\frac{2}{k}\sin\varphi_0 -\frac{2}{k}\sin(kz+\varphi_0)\right], \tag{17b}$$

$$a_{21}^{(1)} = \frac{B}{(2\pi)^2}k[\sin\varphi_0-\sin(kz+\varphi_0)], \tag{17c}$$

$$a_{22}^{(1)} = \frac{B}{(2\pi)^2}[\cos\varphi_0-\cos(kz+\varphi_0)-kz\sin(kz+\varphi_0)]. \tag{17d}$$

If we investigate the procedure above, we find it is very similar to the well-known "drift-kick" numerical method in dealing with such problems [6]. In that method, the whole RFQ section is divided into small sections, and for each section, the field can be seen as constant. As the number of sections increased, the solution will approach to the real solution with very high precision. If we repeat the procedure above, we obtain the results for the second order approximation on B as,

$$x'=[a_{21}^{(0)}+a_{21}^{(1)}+a_{21}^{(2)}]x_0+[a_{22}^{(0)}+a_{22}^{(1)}+a_{22}^{(2)}]x_0' \tag{18}$$

$$x=[a_{11}^{(0)}+a_{11}^{(1)}+a_{11}^{(2)}]x_0+[a_{12}^{(0)}+a_{12}^{(1)}+a_{12}^{(2)}]x_0' \tag{19}$$

where

$$\begin{aligned} a_{11}^{(2)} = \frac{B^2}{8(2\pi)^4}[&12-2k^2z^2-5\cos2\varphi_0-12\cos(kz)+\cos(2kz+2\varphi_0)+4\cos(kz+2\varphi_0)+2kz\sin2\varphi_0 \\ &-4kz\sin(kz)+4kz\sin(kz+2\varphi_0)], \end{aligned} \tag{20a}$$

$$\begin{aligned} a_{12}^{(2)} = \frac{B^2}{24(2\pi)^4}[&36z-2k^2z^3+3z\cos2\varphi_0+12z\cos kz+3z\cos(2kz+2\varphi_0)+12z\cos(kz+2\varphi_0)+\frac{9}{k}\sin2\varphi_0 \\ &-\frac{48}{k}\sin(kz)-\frac{9}{k}\sin(2kz+2\varphi_0)], \end{aligned} \tag{20b}$$

$$a_{21}^{(2)} = \frac{B^2}{4(2\pi)^4}k[2kz+2kz\cos(kz)-2kz\cos(kz+2\varphi_0)-\sin2\varphi_0-4\sin(kz)+\sin(2kz+2\varphi_0)], \tag{20c}$$

$$\begin{aligned} a_{22}^{(2)} = \frac{B^2}{8(2\pi)^4}[&12-2k^2z^2+\cos2\varphi_0-12\cos(kz)-5\cos(2kz+2\varphi_0)+4\cos(kz+2\varphi_0)-4kz\sin(kz) \\ &-2kz\sin(2kz+2\varphi_0)-4kz\sin(kz+2\varphi_0)]. \end{aligned} \tag{20d}$$

In general, we can find the transfer matrix of a RF electric quadrupole is expressed as

$$R=R^{(0)}+R^{(1)}+R^{(2)}+\cdots+R^{(i)}+\cdots , \tag{21}$$

where $R^{(i)}$ is proportional to the i'th power of $B/4\pi^2$. Usually, B is less than 10 for RFQ accelerators.

3 Discussion

From the preceding section we know that the transform matrix is a power series of $B/4\pi^2$, and the number of terms increases with the increase of the power number. It is hard to get its general convergent property in theory. According to the definition of a transform matrix, if the initial coordinates of the particle are (1, 0), we have the particle coordinates at z as

$$\begin{pmatrix} x(z) \\ x'(z) \end{pmatrix} = R\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}. \tag{22}$$

If the initial coordinates of the particle are (0, 1), then the particle coordinates at z are

$$\begin{pmatrix} x(z) \\ x'(z) \end{pmatrix} = R\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}. \tag{23}$$

That means the elements of the transform matrix are the solutions of the motion equations with special initial conditions. In order to verify the validation condition and its precision, we calculated the transform matrix by integrating the motion Eqs. (8) and (9) directly with initial conditions (1, 0) and (0, 1), and compared the results with those calculated from the formulas we obtained in the last section to see where the series can be truncated and its precision.

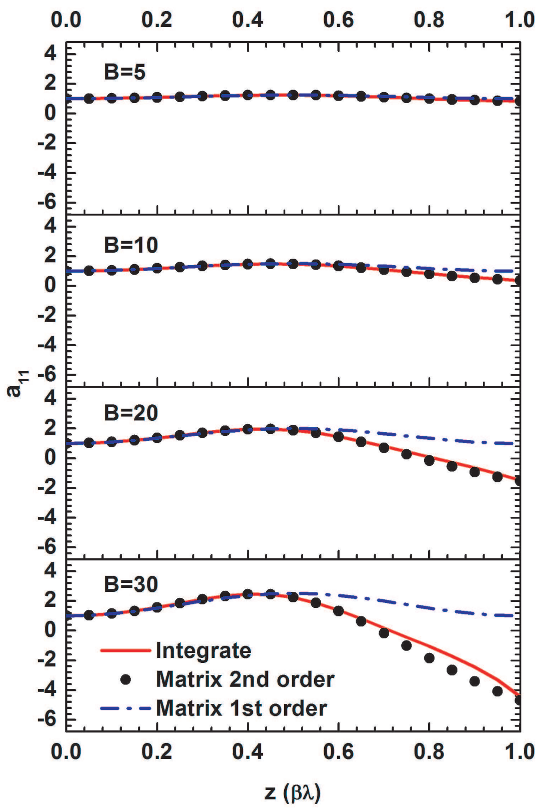

Fig. 2. a_{11} as a function of z, with initial phase 0 degrees. Solid red lines: integration of the motion equation; dash-dotted blue lines: matrix to 1st order; black dots: matrix to 2nd order.

Figure 2 shows a_{11} calculated by motion equation integration and transform matrix formulas to first and second order approximation, where the initial phase is 0 degrees. For $B=5$, the results calculated in three ways agree well for z from zero to $\beta\lambda$. As B increases, the deviation between the 1st order formula and the other two curves becomes larger when z is greater than $0.5\beta\lambda$, while the 2nd order formula and the integration results agree well except for the $B=30$ case. This indicates that the 2nd order formula can approximate the motion equation well for B up to 30 and z less than $\beta\lambda$, and for $z<0.5\beta\lambda$ the 1st order formula is already accurate enough.

Figure 3 shows the four transform matrix elements calculated in three different ways, where B is set to 10, and the RF phase when particles enter the RF electric quadrupole is $-90°$. The second order matrix elements agree with the integration results very well for z less than $\beta\lambda$. For $z<0.5\beta\lambda$ the 1st order formula is a good approximation.

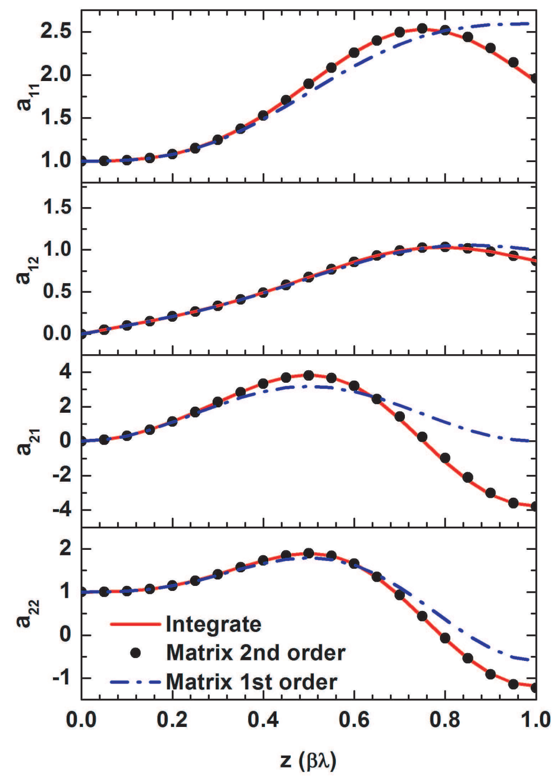

Fig. 3. Results of transform matrix elements ($B=$ 10 and initial phase is $-90°$).

From Eq. (11), we have concluded that the transform matrix which is valid for $z\leqslant 0.5\beta\lambda$ is just needed. From the discussions above we find that the maximum deviation happens at $0.5\beta\lambda$. Since $R(0.5\beta\lambda,\varphi_0,B)$ may be used several times for a long quadrupole, it is better to use the 2nd order form for$R(0.55\beta\lambda,\varphi_0,B)$. However for

$R(z,\varphi_0,B)$ with $z<0.5\beta\lambda$, the 1st order form can be applied for simplicity. Letting $kz=\pi$ and taking it into Eqs. (17) and (20), we obtain the formula of $R(0.55\beta\lambda,\varphi_0,B)$ as follows,

$$R(0.5\beta\lambda,\varphi_0,B)=\begin{pmatrix} a_{\pi,11} & a_{\pi,12} \\ a_{\pi,21} & a_{\pi,22} \end{pmatrix}. \tag{24}$$

where

$$a_{\pi,11} = 1+\frac{B}{(2\pi)^2}(2\cos\varphi_0-\pi\sin\varphi_0)+\frac{B^2}{(2\pi)^4}\left[3-\frac{\pi^2}{4}-\cos2\varphi_0-\frac{\pi}{4}\sin2\varphi_0\right], \tag{25a}$$

$$a_{\pi,12} = \frac{1}{2}\beta\lambda\left[1-\frac{B}{(2\pi)^2}\frac{4}{\pi}\sin\varphi_0+\frac{B^2}{(2\pi)^4}\left(1-\frac{\pi^2}{12}-\frac{1}{4}\cos2\varphi_0\right)\right], \tag{25b}$$

$$a_{\pi,21} = -\frac{2\pi}{\beta\lambda}\left[\frac{B}{(2\pi)^2}2\sin\varphi_0+\frac{B^2}{(2\pi)^4}\frac{\pi}{2}\cos2\varphi_0\right], \tag{25c}$$

$$a_{\pi,22} = 1-\frac{B}{(2\pi)^2}(2\cos\varphi_0+\pi\sin\varphi_0)+\frac{B^2}{(2\pi)^4}\left[3-\frac{\pi^2}{4}-\cos2\varphi_0+\frac{\pi}{4}\sin2\varphi_0\right]. \tag{25d}$$

4 Conclusions

A discrete RF quadrupole cannot be equivalent to a static quadrupole, and its focusing strength is a function of the RF phase when particles enter the quadrupole, so the transverse motion is coupled with the longitudinal motion for RF accelerators which apply discrete RF quadrupoles as transverse focusing elements. The transform matrix of a RF quadrupole is a power series of the focusing parameter B. When the quadrupole length is less than $\beta\lambda$ and B is less than 30, the series up to the 2nd order of B can precisely describe the focusing properties of the RF quadrupole, while if the quadrupole length is less than $0.5\beta\lambda$ and B is less than 30, the series up to the 1st order of B agrees well with the real solution of the motion equation. With a 2nd order transform matrix of a RF quadrupole with length $0.5\beta\lambda$ and 1st order transform matrix of a quadrupole with length less than $0.5\beta\lambda$, the particle motion in an arbitrarily long RF quadrupole can be described.

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