

NLO effects for Ω_{QQQ} baryons in QCD Sum Rules*Ren-Hua Wu(吴仁华)^{1†} Yu-Sheng Zuo(左羽生)^{1‡} Ce Meng(孟策)^{1§}
Yan-Qing Ma(马滢青)^{1,2,3¶} Kuang-Ta Chao(赵光达)^{1,2,3‡}¹School of Physics and State Key Laboratory of Nuclear Physics and Technology, Peking University, Beijing 100871, China²Center for High Energy Physics, Peking University, Beijing 100871, China³Collaborative Innovation Center of Quantum Matter, Beijing 100871, China

Abstract: We study the triply heavy baryons Ω_{QQQ} ($Q = c, b$) in the QCD sum rules by performing the first calculation of the next-to-leading order (NLO) contribution to the perturbative QCD part of the correlation functions. Compared with the leading order (LO) result, the NLO contribution is found to be very important to the Ω_{QQQ} . This is because the NLO not only results in a large correction but also reduces the parameter dependence, making the Borel platform more distinct, especially for the Ω_{bbb} in the $\overline{\text{MS}}$ scheme, where the platform appears only at NLO but not at LO. Particularly, owing to the inclusion of the NLO contribution, the renormalization schemes ($\overline{\text{MS}}$ and On-Shell) dependence and the scale dependence are significantly reduced. Consequently, after including the NLO contribution to the perturbative part in the QCD sum rules, the masses are estimated to be $4.53^{+0.26}_{-0.11}$ GeV for Ω_{ccc} and $14.27^{+0.33}_{-0.32}$ GeV for Ω_{bbb} , where the results are obtained at $\mu = M_B$ with errors including those from the variation of the renormalization scale μ in the range $(0.8 - 1.2)M_B$. A careful study of the μ dependence in a wider range is further performed, which shows that the LO results are very sensitive to the choice of μ whereas the NLO results are considerably better. In addition to the $\mu = M_B$ result, a more stable value, $(4.75-4.80)$ GeV, for the Ω_{ccc} mass is found in the range of $\mu = (1.2 - 2.0)M_B$, which should be viewed as a more relevant prediction in our NLO approach because of μ dependence.

Keywords: baryon, heavy quark, QCD sum rules, NLO correction**DOI:** 10.1088/1674-1137/ac0b3c

I. INTRODUCTION

In recent years, a large number of new hadronic states containing heavy quarks (charm quark c or bottom quark b) have been observed in hadron and e^+e^- colliders [1]. For example, the tetraquark states, pentaquark states, and baryons, which contain two heavy quarks [2-4]. These findings have opened up a new stage for the study of hadron physics and QCD. It is well known that the light flavor baryons are composed of three light quarks (u, d, s), and all kinds of light flavor baryons from $\Delta^{++}(uuu)$ to $\Omega^-(sss)$ have been observed for many decades. For the heavy baryons, the singly charmed and bottom baryons (containing one heavy quark c or b and two light quarks)

have already formed a subfamily within heavy hadrons in recent years. Remarkably, the doubly charmed baryon, $\Xi_{cc}^{++}(ccu)$, which contains two charm quarks and one light quark, was discovered in 2017 [5], and more of its kind are expected to be found soon. The discovery of the doubly charmed baryon is an important event, and it may indicate that the whole family of baryons with all flavors (u, d, s, c, b) may be found in the not far future. Here, the last member of the baryon family, i.e., the triply heavy baryons, which are composed of three heavy quarks c or b , are yet to be discovered.

The Ω_{QQQ} baryons are made of three identical heavy quarks ($Q = c, b$). Because of their special properties in the baryon family, the Ω_{QQQ} baryons have been studied

Received 9 June 2021; Accepted 15 June 2021; Published online 30 July 2021

* Supported in part by the National Natural Science Foundation of China (11875071, 11975029) and the National Key Research and Development Program of China (2020YFA0406400)

† E-mail: renhuawu@pku.edu.cn

‡ E-mail: 1801210125@pku.edu.cn

§ E-mail: mengce75@pku.edu.cn

¶ E-mail: yqma@pku.edu.cn

‡ E-mail: ktchao@pku.edu.cn



Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI. Article funded by SCOAP³ and published under licence by Chinese Physical Society and the Institute of High Energy Physics of the Chinese Academy of Sciences and the Institute of Modern Physics of the Chinese Academy of Sciences and IOP Publishing Ltd

extensively for their productions [6-12], weak decays [13-15], masses [16-43] and so on. Particularly, the masses of Ω_{QQQ} baryons have been evaluated using various approaches, including lattice QCD [16-22], QCD sum rules [23-25], potential models [26-37], the Fadeev equation [38-41], and the Regge trajectories [42, 43]. The predicted values for the masses of Ω_{QQQ} baryons are listed in Table 5. As shown, the predicted masses are in a wide range, and different approaches produce different results. Therefore, it is still necessary to examine the Ω_{QQQ} baryons.

In this paper, we present the study of Ω_{QQQ} baryon using the QCD sum rule [44-46] approach, which is known to be a powerful tool to evaluate hadron properties [47-50]. In this approach, one starts at short distances and then moves to long distances using the operator product expansion (OPE) for a given quark current's correlation function in QCD. The first term in the OPE is the dimensionless ($d=0$) identity operator, which represents the perturbative QCD part, and then, as power corrections, the higher dimensional operators ($d=4, 6, \dots$) with vacuum condensations emerge, which represent the nonperturbative (confinement) contribution. Principally,

because of the power suppression $\left(\frac{\Lambda_{\text{QCD}}^2}{-q^2} \ll 1\right)$ at small distances (large $(-q^2)$), the high dimensional operator contributions should decrease as the power increases; one may only need to consider a few important terms in the OPE. Moreover, for the first (Identity) term in the OPE, i.e., the perturbative QCD term, one needs to consider not only the leading-order (LO) but also, at least, the next-to-leading order (NLO) $O(\alpha_s)$ contributions (α_s being the strong coupling constant), because the latter may lead to substantial corrections. Additionally, the $O(\alpha_s)$ corrections to the coefficients of the power-suppressed condensation terms may also need to be considered. Practically, for the heavy Ω_{QQQ} system, the most important contributions in the OPE are the perturbative term C_1 and gluon condensation term $C_{GG}\langle g_s^2 \hat{G}\hat{G} \rangle$, where C_1 and C_{GG} can be calculated perturbatively. As a good approximation, it may be necessary to consider the LO of C_1 , which is the most important contribution, the NLO of C_1 , which gives large corrections, and the LO of C_{GG} , which is the same order of magnitude as the former, but neglects other contributions. In fact, the importance of including the NLO contribution of C_1 has been emphasized in many studies, e.g., for the proton (uud) [51, 52], the singly heavy baryon [53], and the doubly heavy baryon Ξ_{cc}^{++} [54]. Our previous work [54] analyzed the NLO effect thoroughly for the doubly heavy baryon Ξ_{cc}^{++} and determined that the NLO correction is sizable for Ξ_{cc}^{++} and cannot be ignored in QCD sum rules. Accordingly, we expect that the NLO correction also produces a sizable contribution to triply heavy baryons (QQQ). Presently, no work has been conducted at the NLO level for triply heavy baryons. However, for some leading order (LO) results of the

Ω_{QQQ} [23-25], there are significant differences among different studies, as shown in the preceding paragraph. To reduce the uncertainties at the LO, it may be necessary to perform the NLO calculations in QCD sum rules. With the inclusion of the NLO contribution, the result should be substantially improved. Moreover, it is worthwhile to emphasize that there are significant differences between the fully heavy baryons and other baryons that contain light quarks. For the former, the most important nonperturbative contribution comes from the gluon condensation $\langle GG \rangle$, whereas for the latter, the light quark condensation $\langle \bar{q}q \rangle$ makes the important contribution. This point will be embodied in our calculation for Ω_{QQQ} in the QCD sum rules.

The rest of the paper is organized as follows. In Sec. II, the sum rules for the calculation of the mass of Ω_{QQQ} are presented. In Sec. III, we introduce the methods and procedures for the calculation of the coefficients C_1 and C_{GG} . The phenomenology results and discussions are presented in Sec. IV.

II. QCD SUM RULES

For the S -wave triply heavy quark (QQQ) system, owing to Fermi-Dirac statistics [25, 55], there only exists the $J^P = \frac{3}{2}^+$ baryon ground state Ω_{QQQ} ($Q=c, b$), with the corresponding current J_μ [23, 25, 55],

$$J_\mu = \epsilon^{abc} (Q_a^T \hat{C} \gamma^\mu Q_b) Q_c, \quad (1)$$

where a, b , and c denote the color indices, and \hat{C} is the charge-conjugation matrix. Because there is no QQQ ground state with $J = \frac{1}{2}$, it is not necessary to worry about the pollution from the $J = \frac{1}{2}$ state in the analysis.

In the QCD sum rules, we begin with the two-point correlation function

$$\Pi_{\mu\nu}(q^2) = i \int d^4x e^{iq \cdot x} \langle \Omega | T [J_\mu(x) \bar{J}_\nu(0)] | \Omega \rangle, \quad (2)$$

where Ω represents the QCD vacuum. According to the Lorentz covariance, the matrix element of the current can be written as

$$\langle \Omega | J_\mu(0) | H(q, s) \rangle = \sqrt{\lambda} U_\mu(q, s), \quad (3)$$

where $H(q, s)$ denotes the ground state baryon with mass M_H , momentum q , and spin s ; λ is the pole residue for H ; and $U_\mu(q, s)$ is the corresponding Rarita-Schwinger spinor, which satisfies the relation

$$\sum_s U_\mu(q, s) \bar{U}_\nu(q, s) = (q + M_H) \left(-g_{\mu\nu} + \frac{\gamma_\mu \gamma_\nu}{3} + \frac{2q_\mu q_\nu}{3M_H^2} - \frac{q_\mu \gamma_\nu - \gamma_\mu q_\nu}{3M_H} \right). \quad (4)$$

Correlation function $\Pi_{\mu\nu}$ can then be written in the form

$$\begin{aligned} \Pi_{\mu\nu}(q^2) = M \left(-g_{\mu\nu}\Pi_1(q^2) + \frac{\gamma_\mu\gamma_\nu}{3}\Pi_2(q^2) + \frac{2q_\mu q_\nu}{3M^2}\Pi_3(q^2) \right. \\ \left. - \frac{q_\mu\gamma_\nu - \gamma_\mu q_\nu}{3M}\Pi_4(q^2) \right) + q \left(-g_{\mu\nu}\Pi_5(q^2) \right. \\ \left. + \frac{\gamma_\mu\gamma_\nu}{3}\Pi_6(q^2) + \frac{2q_\mu q_\nu}{3M^2}\Pi_7(q^2) - \frac{q_\mu\gamma_\nu - \gamma_\mu q_\nu}{3M}\Pi_8(q^2) \right). \end{aligned} \quad (5)$$

In this paper, we choose $\Pi_1(q^2)$ for the calculation to obtain mass M_H of the ground state in the QCD sum rules. For convenience, in the following, we use $\Pi(q^2)$ to denote $\Pi_1(q^2)$.

However, correlation function $\Pi(q^2)$ can be related to the phenomenological spectrum by the Källén-Lehmann representation

$$\Pi(q^2) = - \int ds \frac{\rho(s)}{q^2 - s + i\epsilon}, \quad (6)$$

where $\rho(s)$ is the spectrum density, which contains information about all resonances and the continuum. Taking the narrow resonance approximation for the physical ground state, we may assume the spectrum density to consist of a single pole and a continuum spectrum, where all excited states are included in the continuum spectrum

$$\rho(s) = \lambda\delta(s - M_H^2) + \rho_{\text{cont}}(s), \quad (7)$$

where $\rho_{\text{cont}}(s)$ denotes the continuum spectral density.

However, in the region where $-q^2 = Q^2 \gg \Lambda_{\text{QCD}}^2$, correlation function $\Pi(q^2)$ can be calculated using the OPE, which reads

$$\Pi(q^2) = C_1(q^2) + \sum_i C_i(q^2)\langle O_i \rangle, \quad (8)$$

where C_1 and C_i are the perturbatively calculable Wilson coefficients for perturbative term $\langle \Omega|1|\Omega \rangle = 1$ and vacuum condensation $\langle O_i \rangle = \langle \Omega|O_i|\Omega \rangle$, respectively. The relative importance of the vacuum condensation is power suppressed by the dimension of operator O_i . In our calculations, we only maintain the vacuum condensations up to dimension $d = 4$, which gives the approximation expression of the OPE as

$$\Pi(q^2) = C_1(q^2) + C_{GG}(q^2)\langle g_s^2\hat{G}\hat{G} \rangle, \quad (9)$$

where $\langle g_s^2\hat{G}\hat{G} \rangle$ denotes the gluon-gluon (GG) condensation. Here, the contributions of higher dimensional condensates are expected to be small because of the power suppressions. Particularly, the $d = 6$ term, $\langle g_s^3\hat{G}\hat{G}\hat{G} \rangle$, is

neglected, and this is illustrated in a recent work [25], where the contribution of this operator is shown to be negligible.

According to Eq. (6), one can relate the physical spectrum density to the imaginary part of $\Pi(q^2)$ in Eq. (9) using the dispersion relation, which gives

$$\begin{aligned} \Pi(q^2) &= \int ds \frac{\rho(s)}{s - q^2 - i\epsilon} \\ &= \frac{1}{\pi} \int_{s_{\text{th}}}^{\infty} ds \frac{\text{Im}C_1(s) + \text{Im}C_{GG}(s)\langle g_s^2\hat{G}\hat{G} \rangle}{s - q^2 - i\epsilon}, \end{aligned} \quad (10)$$

where $s_{\text{th}} = 9m_Q^2$ is the QCD threshold for the QQQ system, and the integral in the second line is assumed to be convergent.

To extract the mass of the ground state, we first employ the quark-hadron duality [47, 56-58] and the Borel transformation, and obtain a sum rule for $\Pi(q^2)$,

$$\begin{aligned} \lambda e^{-\frac{M_H^2}{M_B^2}} &= \int_{s_{\text{th}}}^{s_0} ds \frac{1}{\pi} \text{Im}C_1(s) e^{-\frac{s}{M_B^2}} \\ &+ \int_{s_{\text{th}}}^{\infty} ds \frac{1}{\pi} \text{Im}C_{GG}(s) e^{-\frac{s}{M_B^2}} \langle g_s^2\hat{G}\hat{G} \rangle, \end{aligned} \quad (11)$$

where s_0 and M_B are the continuum threshold and Borel parameters, respectively, which are introduced here owing to the quark-hadron duality and Borel transformation. Differentiating both sides of Eq. (11) with respect to $-\frac{1}{M_B^2}$, we obtain

$$\begin{aligned} \lambda M_H^2 e^{-\frac{M_H^2}{M_B^2}} &= \int_{s_{\text{th}}}^{s_0} ds \frac{1}{\pi} \text{Im}C_1(s) e^{-\frac{s}{M_B^2}} s \\ &+ \int_{s_{\text{th}}}^{\infty} ds \frac{1}{\pi} \text{Im}C_{GG}(s) e^{-\frac{s}{M_B^2}} s \langle g_s^2\hat{G}\hat{G} \rangle. \end{aligned} \quad (12)$$

Finally, we can solve M_H using Eq. (11) and (12),

$$M_H^2 = \frac{\int_{s_{\text{th}}}^{s_0} ds \rho_1(s) e^{-\frac{s}{M_B^2}} s + \int_{s_{\text{th}}}^{\infty} ds \rho_{GG}(s) e^{-\frac{s}{M_B^2}} s \langle g_s^2\hat{G}\hat{G} \rangle}{\int_{s_{\text{th}}}^{s_0} ds \rho_1(s) e^{-\frac{s}{M_B^2}} + \int_{s_{\text{th}}}^{\infty} ds \rho_{GG}(s) e^{-\frac{s}{M_B^2}} \langle g_s^2\hat{G}\hat{G} \rangle}, \quad (13)$$

where $\rho_1 = \frac{1}{\pi} \text{Im}C_1$ and $\rho_{GG} = \frac{1}{\pi} \text{Im}C_{GG}$.

III. CALCULATIONS OF C_1 AND C_{GG} FOR QQQ BARYON

In QCD sum rules, there are two kinds of expansions, the OPE and perturbative expansion in α_s . For the OPE, we consider the most important contributions, i.e., per-

turbative term C_1 and the GG condensation, $C_{GG}\langle g_s^2 \hat{G}\hat{G} \rangle$, because other higher dimensional operators are power suppressed. According to Eq. (13), we need the imaginary parts of C_1 and C_{GG} , which can be calculated perturbatively.

We use FeynArts [59, 60] to generate the Feynman diagrams and amplitudes of C_1 and C_{GG} . The LO and NLO Feynman diagrams are shown in Fig. 1 and Fig. 2, respectively.

In the Ω_{QQQ} system, the three quarks have the same flavor, thus, some diagrams are similar and not shown again. For example, there are three cases in the second diagram of C_1 -NLO in Fig. 2, which denotes the cases with one gluon exchange between any two heavy quarks.

The calculation procedures for C_1 and C_{GG} are summarized below:

1. We use FeynCalc [61, 62] to simplify the spinor structures of the Feynman amplitudes with the Naive- γ_5 scheme.

2. We use Reduze [63] to reduce all the loop integrals \mathbf{I} to the linear combination $\mathbf{I} = \sum a_i \tilde{I}_i$, where \tilde{I}_i 's are the so-called master integrals (MIs). Furthermore, a_i 's are the reduced coefficients, which are rational functions in the Mandelstam invariant (s), quark mass (m_Q), and time-space dimension ($D = 4 - 2\epsilon$) in this work.

3. We set up differential equations for MIs [64-67] and solve them numerically, with boundary conditions obtained via the auxiliary mass flow method [68]. (The expressions of the MIs are the series-expansion forms, i.e.

$$\tilde{I}(s, r, \epsilon) = s^{a+b\epsilon} \sum_{n,m} \left(\sum_i c_{mni} \epsilon^i \right) \text{Log}(r)^m r^n, \text{ where } r = m_Q^2/s$$

and c_{mni} 's are float numbers rather than rational numbers. The expressions of the Wilson coefficients C_1 and C_{GG} are complex and not presented in this paper. These results will be shared in auxiliary files.)

4. Renormalization. There are no infrared divergences in the NLO amplitude of C_1 . After performing the wave-function and mass renormalization of the quarks (m_Q is renormalized in either the $\overline{\text{MS}}$ or On-Shell scheme), the remaining ultraviolet divergences can be removed by the operator renormalization of current J_μ . We renormalize the current in the $\overline{\text{MS}}$ scheme, and find the renormalization constant up to the NLO level as

$$Z_O = 1 + \frac{\alpha_s}{6\pi} \left(\frac{1}{\epsilon} + \text{Log}(4\pi) - \gamma_E \right) \quad (14)$$

IV. PHENOMENOLOGY

In our numerical analysis, we choose the following parameters [54, 69-73].

$$\begin{aligned} m_c^{\overline{\text{MS}}}(m_c) &= 1.28 \pm 0.03 \text{ GeV} \\ m_c^{\text{OS}} &= 1.46 \pm 0.07 \text{ GeV} \\ m_b^{\overline{\text{MS}}}(m_b) &= 4.18 \pm 0.03 \text{ GeV} \\ m_b^{\text{OS}} &= 4.65 \pm 0.05 \text{ GeV} \\ \langle g_s^2 \hat{G}\hat{G} \rangle &= 4\pi^2(0.037 \pm 0.015) \text{ GeV}^4 \\ \alpha_s(m_Z = 91.1876 \text{ GeV}) &= 0.1181 \end{aligned} \quad (15)$$

It is worth emphasizing that $\alpha_s(\mu)$ and the heavy quark mass, $m_Q^{\overline{\text{MS}}}(\mu)$, are obtained through two-loop run-

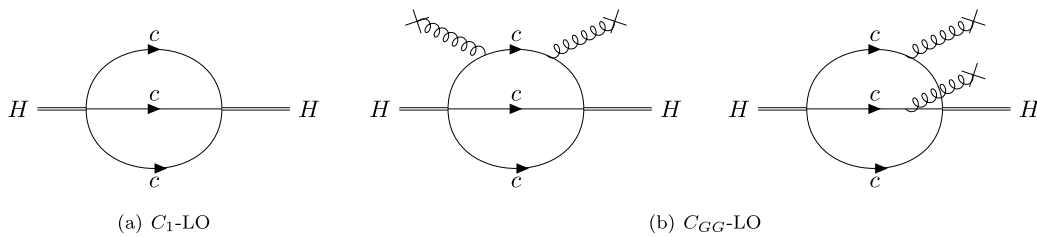


Fig. 1. LO Feynman Diagrams of C_1 and C_{GG} . H denotes the interpolating current.

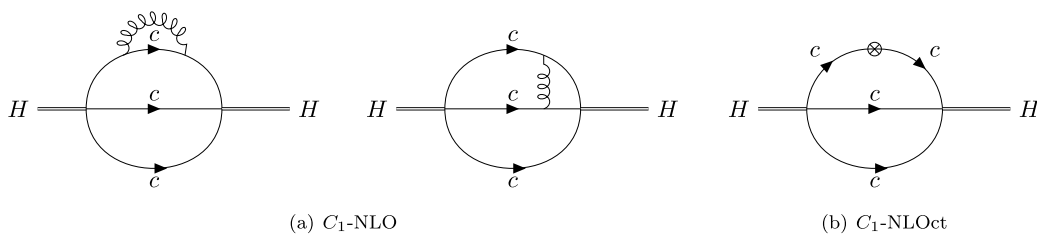


Fig. 2. NLO and counter term Feynman Diagrams of C_1 . H denotes the interpolating current.

ning [54]. As a typical and tentative choice, we set the renormalization scale, μ , to be equal to the Borel parameter, M_B in our phenomenological analysis [44, 74] (we will discuss the renormalization scale dependence of the results by setting μ with other values later). Moreover, the On-Shell (OS) masses m_c^{OS} and m_b^{OS} are extracted from the QCD sum rules analysis of the J/ψ and $\Upsilon(1S)$ spectrums, respectively, in which the mass renormalization scheme and truncation order of α_s are the same as ours.

According to Eq. (13), the numerical result of M_H depends on two parameters: s_0 and M_B . Principally, M_H as a physical mass, is independent of any artificial parameters. Therefore, a credible result should be obtained in an appropriate region of these two parameters, where M_H is dependent weakly on M_B and s_0 . Additionally, the choices of M_B and s_0 should ensure the validity of the OPE and dominance of the ground-state pole contribution, which will constrain the two parameters within a suitable parameter space, the so-called "Borel window". Within the Borel window, we find the so-called "Borel platform", in which M_H weakly depends on s_0 and M_B .

To search for the Borel window, we define the relative contributions of the condensation and continuum as

$$r_{GG} = \frac{\langle g_s^2 \hat{G} \hat{G} \rangle \int_{s_{\text{th}}}^{\infty} ds \rho_{GG}(s) e^{-\frac{s}{M_B^2}}}{\int_{s_{\text{th}}}^{\infty} ds \rho_1(s) e^{-\frac{s}{M_B^2}}}$$

$$r_{\text{cont}} = \frac{\int_{s_0}^{\infty} ds \rho_1(s) e^{-\frac{s}{M_B^2}}}{\int_{s_{\text{th}}}^{\infty} ds \rho_1(s) e^{-\frac{s}{M_B^2}}} \quad (16)$$

and impose the following constraints:

$$|r_{GG}| \leq 30\%, \quad |r_{\text{cont}}| \leq 30\%. \quad (17)$$

The two constraints guarantee the validity of the OPE and the ground-state contribution dominance, separately. To find the Borel platform, we first search for the point on which the parameter dependence of M_H is weakest within the Borel window. In addition to the conditions

given in (17), we also impose the following constrain on s_0 :

$$s_0 < (M_H + 1 \text{ GeV})^2, \quad (18)$$

since s_0 roughly denotes the energy scale where the continuum spectrum begins and the energy level spacing of a heavy hadron system is usually smaller than 1 GeV. More explicitly, we choose the variables as $x = s_0$ and $y = M_B^2$ and define the function describing the flatness degree as

$$\Delta(x, y) = \left(\frac{\partial M_H}{\partial x} \right)^2 + \left(\frac{\partial M_H}{\partial y} \right)^2. \quad (19)$$

Thus, minimizing the function $\Delta(x, y)$ within the Borel window and with the constrain (18), we obtain the point (x_0, y_0) , which will be used to evaluate the central value of M_H . We then vary the values of s_0 and M_B^2 around the point (x_0, y_0) up to a 10% magnitude to estimate the errors of M_H . It should be emphasized that the central point (x_0, y_0) may lie on the margin of the Borel window in some cases, therefore, the parameter space used to estimate the errors of M_H may exceed the Borel window; additionally, the upper and lower errors are usually not symmetric.

A. Ω_{ccc} system

The results for Ω_{ccc}^{++} in the $\overline{\text{MS}}$ and OS schemes are listed in Table 1. The Borel platform curves, which show the parameter dependence of M_H on s_0 and M_B^2 , are shown in Fig. 3. In all the curves, the dot corresponds to the central point (x_0, y_0) , and the shadows denote the Borel window determined by Eq. (17)

First, from Fig. 3, we observe that there exists a perfect Borel platform in the $\overline{\text{MS}}$ scheme. However, there is no decent Borel platform in the OS scheme, thus, the result in the OS scheme is not good. Therefore, we consider the result in the $\overline{\text{MS}}$ scheme as our prediction for the mass of Ω_{ccc}^{++} .

Secondly, although the OS result is not good, we can still observe the distinct NLO effects in the reduction of the scheme dependence by the comparison between the $\overline{\text{MS}}$ and OS schemes. In fact, at the LO a big gap is ob-

Table 1. LO and NLO results for the mass of Ω_{ccc}^{++} in $\overline{\text{MS}}$ and On-Shell schemes. Here, the errors for M_H are from s_0, M_B , the charm quark mass, and the renormalization scale μ with $\mu = kM_B$ and $k \in (0.8, 1.2)$ (the central values correspond to $\mu = M_B$).

Order	M_H/GeV	s_0/GeV^2	M_B^2/GeV^2	Error from s_0 and M_B^2	Error from m_Q	Error from μ
LO($\overline{\text{MS}}$)	$4.39_{-0.22}^{+0.24}$	$29(\pm 10\%)$	$1.75(\pm 10\%)$	$+0.02$ -0.03	$+0.10$ -0.10	$+0.22$ -0.19
NLO($\overline{\text{MS}}$)	$4.53_{-0.11}^{+0.26}$	$26(\pm 10\%)$	$1.20(\pm 10\%)$	$+0.02$ -0.04	$+0.09$ -0.10	$+0.24$ -0.00
LO(OS)	$4.51_{-0.23}^{+0.19}$	$21(\pm 10\%)$	$0.40(\pm 10\%)$	$+0.06$ -0.15	$+0.18$ -0.18	
NLO(OS)	$4.55_{-0.18}^{+0.13}$	$22(\pm 10\%)$	$0.90(\pm 10\%)$	$+0.07$ -0.12	$+0.11$ -0.14	

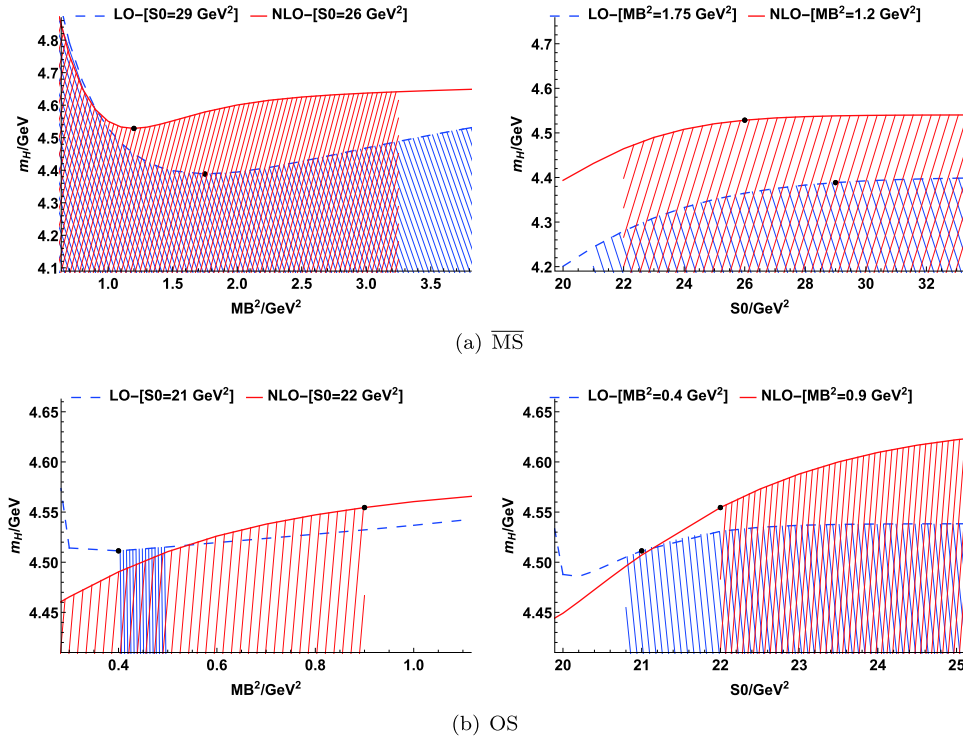


Fig. 3. (color online) Borel platform curves for the Ω_{ccc}^{++} in $\overline{\text{MS}}$ and On-Shell schemes.

served between the two schemes results, whereas the $\overline{\text{MS}}$ result increases markedly at the NLO, which brings the results of the two schemes close to each other at the NLO. Furthermore, the quark mass dependence of the results is also reduced at the NLO. Particularly, the NLO contribution leads to a noteworthy and significant correction of the LO result, and cannot be ignored in the QCD sum rules for the triply charmed baryons.

Furthermore, as already mentioned, we will evaluate the renormalization scale μ dependence of the LO and NLO results in the $\overline{\text{MS}}$ scheme. The results obtained with $\mu = M_B$ are listed in Table 1. We then study the results with different μ ($\mu = k M_B$) and $k \in (0.8, 2.0)$, which are presented in Fig. 4 and Table 2. The range of μ is chosen with the requirement that the Borel platform can be achieved and the perturbative expansion is under good control. From Fig. 4 it can be clearly observed that the μ dependence is significantly reduced for the NLO result, whereas the LO result is very sensitive to the choice of μ . Additionally, it is worth noting that the obtained NLO mass of Ω_{ccc}^{++} is 4.75–4.80 GeV for a rather wide range of $\mu = (1.2 - 2.0)M_B$. Considering all important factors, like μ dependence and perturbative convergence, 4.75–4.80 GeV is a more credible prediction for the Ω_{ccc} mass. Interestingly enough, this value is in good agreement with the lattice QCD result.

It is worth mentioning that our result includes errors due to the variation of the renormalization scale, as compared with previous LO results [23, 25] (4.67 ± 0.15 GeV

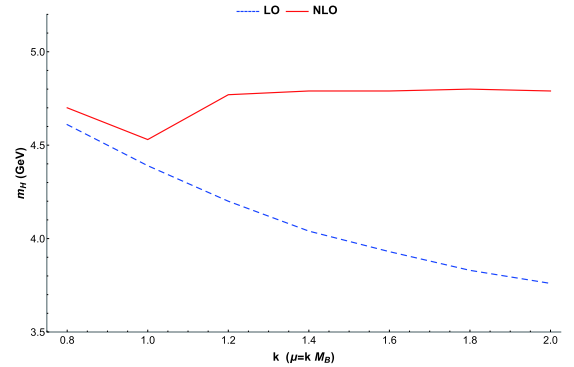


Fig. 4. (color online) Dependence on renormalization scale μ in $\overline{\text{MS}}$ scheme for Ω_{ccc}^{++} .

Table 2. $\overline{\text{MS}}$ mass of Ω_{ccc} with different renormalization scale $\mu = k M_B$.

k ($\mu = k M_B$)	LO/GeV	NLO/GeV
0.8	4.65	4.70
1.0	4.39	4.53
1.2	4.20	4.77
1.4	4.05	4.79
1.6	3.93	4.79
1.8	3.83	4.80
2.0	3.76	4.79

and 4.81 ± 0.10 GeV, respectively, in which the scale dependence is not considered). Regarding the central val-

ues, the differences between our LO results and those in [23, 25] are mainly caused by the choice of input parameters and determination of the Borel platform. For example, we can reproduce the result in [25] using the parameters therein. In Table 5, we list the predictions for the mass of Ω_{ccc}^{++} in various approaches.

B. Ω_{bbb} system

For the Ω_{bbb}^- , the results in the $\overline{\text{MS}}$ and OS schemes are listed in Table 3, and the Borel platform curves are shown in Fig. 5.

With these figures, we can make similar conclusions as illustrated in the ccc section. For instance, there are no very stable Borel platforms in the OS scheme at both the LO and NLO. Furthermore, there is a distinct difference between the bbb and ccc systems. This can be observed in Fig. 5, where there is no Borel platform at the LO in the $\overline{\text{MS}}$ scheme; however, there appears a clear platform after including the NLO contribution. For the ccc , there are

platforms at both the LO and NLO. This indicates that for the bbb the NLO contribution from the perturbative term C_1 is crucial to the formation of a stable Borel platform in the QCD Sum Rules.

We further evaluate the renormalization scale μ dependence of the Ω_{bbb}^- mass in a wider range of $\mu = (0.8 - 1.4)M_B$ ($\mu > 1.4M_B$ is excluded owing to the absence of the Borel platform), and the results are shown in Fig. 6 and Table 4. It can be observed that the scale dependence is obviously weaker at the NLO than at the LO. Nevertheless, at the NLO, the uncertainty for the Ω_{bbb}^- mass due to the scale dependence is still not small. This implies that all estimates of the masses at the LO in the QCD sum rules must suffer from large uncertainties caused by the scale dependence, and can be improved by including the NLO contributions.

Our result for the Ω_{bbb}^- mass at the LO ($13.97^{+0.54}_{-0.50}$ GeV) is comparable with previous works in QCD sum rules at the LO [23, 25] (13.28 ± 0.10 GeV and

Table 3. LO and NLO results for the mass of Ω_{bbb}^- in $\overline{\text{MS}}$ and On-Shell schemes. Here, the errors for M_H are from s_0, M_B , the bottom quark mass, and renormalization scale μ with $\mu = kM_B$ and $k \in (0.8, 1.2)$ (the central values correspond to $\mu = M_B$).

Order	M_H/GeV	s_0/GeV^2	M_B^2/GeV^2	Error from s_0 and M_B^2	Error from m_Q	Error from μ
LO($\overline{\text{MS}}$)	$13.97^{+0.54}_{-0.50}$	$224(\pm 10\%)$	$17.00(\pm 10\%)$	+0.23 -0.36	+0.06 -0.06	+0.48 -0.34
NLO($\overline{\text{MS}}$)	$14.27^{+0.33}_{-0.32}$	$232(\pm 10\%)$	$10.00(\pm 10\%)$	+0.12 -0.25	+0.08 -0.08	+0.30 -0.19
LO(OS)	$14.00^{+0.15}_{-0.15}$	$197(\pm 10\%)$	$0.40(\pm 10\%)$	+0.01 -0.01	+0.15 -0.15	
NLO(OS)	$14.06^{+0.10}_{-0.14}$	$200(\pm 10\%)$	$1.75(\pm 10\%)$	+0.07 -0.07	+0.07 -0.12	

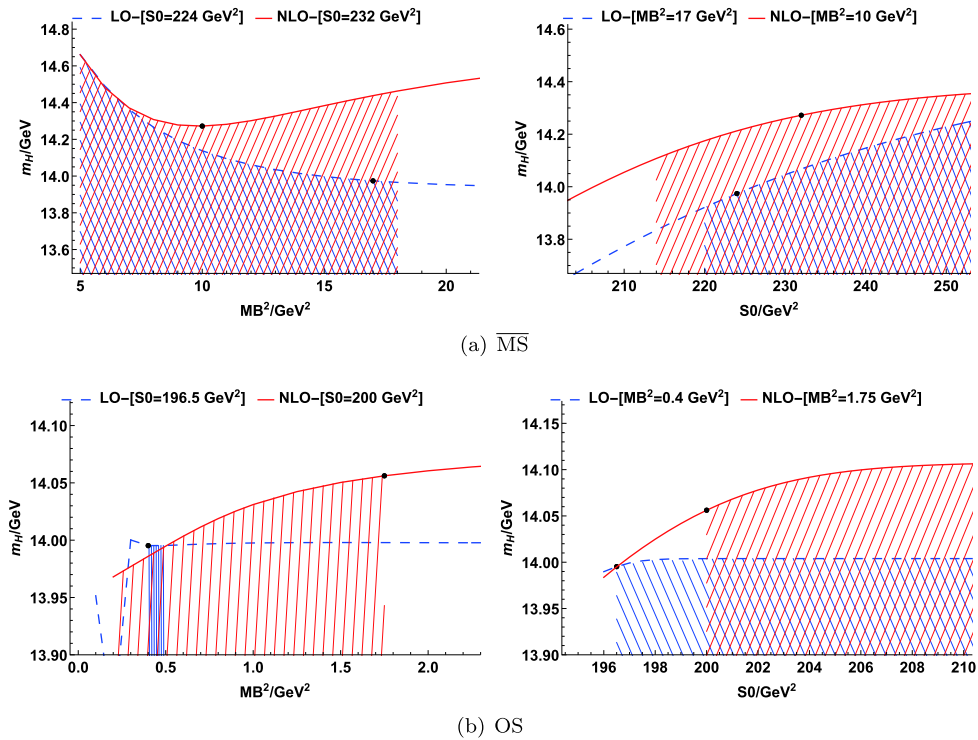
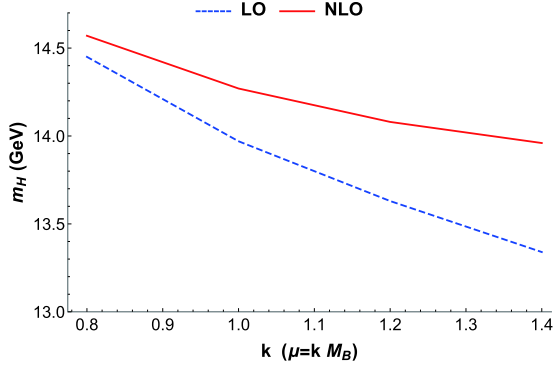


Fig. 5. (color online) Borel platform curves for the Ω_{bbb}^- in $\overline{\text{MS}}$ and On-Shell schemes.

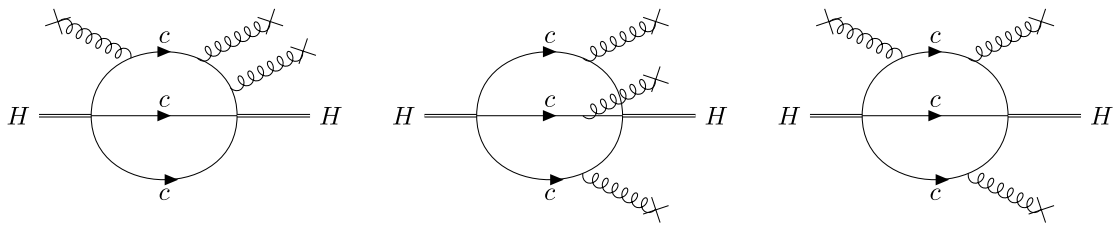
Table 4. $\overline{\text{MS}}$ mass of Ω_{bbb} with different renormalization scale $\mu = kM_B$.

$k (\mu = k M_B)$	LO/GeV	NLO/GeV
0.8	14.45	14.57
1.0	13.97	14.27
1.2	13.63	14.08
1.4	13.34	13.96

**Fig. 6.** (color online) Dependence on renormalization scale μ in $\overline{\text{MS}}$ scheme for Ω_{bbb}^- .**Table 5.** Predicted masses of Ω_{ccc} and Ω_{bbb} in different approaches. Here, the central values of our results are obtained with the renormalization scale $\mu = M_B$. (With $\mu = (1.2-2.0)M_B$ the Ω_{ccc} mass is calculated as 4.75–4.80 GeV, see Table 2 and Fig. 4.)*

Models	Ω_{ccc}/GeV	Ω_{bbb}/GeV
Lattic QCD [16-22]	4.7~4.8	14.36~14.37
QCD Sum Rules [23-25]	4.6~5.0	13.28~14.83
Various potential models [14, 26-37]	4.76~4.90	14.27~14.83
Faddeev equation [38-41]	4.76~5.00	14.23~14.57
Regge trajectories [42, 43]	4.834	14.788
This work	$4.53^{+0.26}_{-0.11}$ 4.75~4.80 GeV*	$14.27^{+0.33}_{-0.32}$

14.43 ± 0.09 GeV respectively), which is similar to the case of Ω_{ccc}^{++} . Theoretical uncertainties are significantly reduced after considering the NLO contribution. We then obtain $14.27^{+0.33}_{-0.32}$ GeV at the NLO in the $\overline{\text{MS}}$ scheme. The

**Fig. 7.** Part contribution of C_{GGG} -LO. H denotes the interpolating current.

results obtained in various approaches are listed in Table 5, all of which are comparable with our results.

C. Discussion of GGG condensate contributions

In this work, we also evaluate the GGG condensate contributions, and find that they are very small; thus, they do not affect the phenomenological results.

The leading order contributions of the GGG condensate come from two parts: one is from the diagrams shown in Fig. 7 and the other is from the higher order of the gluons background field in the C_{GG} -LO of Fig. 1, i.e., $\langle GGG \rangle$ that is related to the $\langle A(x)A(y) \rangle$ expansion. (Similar to the results of C_1 and C_{GG} , the results of C_{GGG} will be shared by auxiliary files). Choosing $\langle g_s^3 \hat{G}\hat{G}\hat{G} \rangle = 0.054 \pm 0.014 \text{ GeV}^6$ [25, 45-47], similar to [25], we find the ratios of different condensate contributions near the Borel platform as

$$\int_{s_{\text{th}}}^{s_0} ds \rho_1(s) e^{-\frac{s}{M_B^2}} : \int_{s_{\text{th}}}^{\infty} ds \rho_{GG}(s) \langle g_s^2 \hat{G}\hat{G} \rangle e^{-\frac{s}{M_B^2}} : \\ \int_{s_{\text{th}}}^{\infty} ds \rho_{GGG}(s) \langle g_s^3 \hat{G}\hat{G}\hat{G} \rangle e^{-\frac{s}{M_B^2}} \simeq 1 : \mathcal{O}(10^{-2}) : \mathcal{O}(10^{-4}) \quad (20)$$

Even if we use a larger value of $\langle g_s^3 \hat{G}\hat{G}\hat{G} \rangle = 0.52 \pm 0.10 \text{ GeV}^6$ as chosen in [75-78], we can still observe that this $d = 6$ GGG condensate contributions are still too small to affect the phenomenological results. This conclusion is consistent with the findings in Ref. [25]. Therefore, according to our study, the GGG condensate contributions can be ignored.

V. SUMMARY

In the study of hadron physics, the QCD sum rule is known to be a powerful tool to evaluate hadron properties. For the triply heavy baryons QQQ ($Q = c, b$), previous works only dealt with the LO perturbative QCD calculation. The absence of the higher order QCD corrections may lead to large theoretical uncertainties in the QCD sum rules. In this paper, for the first time, we calculate the NLO QCD contribution of the QQQ system by considering the NLO correction to the perturbative term of C_1 in the OPE. Additionally, the non perturbative con-

tribution is embodied by the $d = 4$ gluon-gluon condensate. We also introduce adequate constraints to ensure the ground state dominance and power suppressions within the Borel windows.

Because the main theoretical uncertainty of the LO result is from the absence of the NLO correction of C_1 in the QCD sum rules, the inclusion of the NLO contribution plays important roles theoretically and phenomenologically. By comparing the NLO result with the LO one, we may draw the following conclusions for the Ω_{QQQ} system. First, the correction from the NLO contribution is sizable ($\Delta M_H = M_H^{\text{NLO}} - M_H^{\text{LO}}$ can be as large as $0.2 \sim 0.3$ GeV or even larger). Secondly, after considering the NLO contribution, the parameters dependence of the results is reduced and the Borel platform becomes more distinct, especially for the bbb system in the $\overline{\text{MS}}$ scheme. Thirdly, the dependences on the renormalization schemes and quark masses are significantly improved for the Ω_{ccc} .

By including the NLO contribution of the perturbative part in the QCD sum rules, we find the masses to be $4.53^{+0.26}_{-0.11}$ GeV for Ω_{ccc} and $14.27^{+0.33}_{-0.32}$ GeV for Ω_{bbb} , where the results are obtained at $\mu = M_B$ with errors including that from the variation of the renormalization scale μ in the range $(0.8 - 1.2)M_B$. Further study for the μ dependence in a wider range shows that the LO results are very sensitive to the choice of μ , and therefore have large uncertainties, whereas the NLO results are considerably better. Aside from the masses given above at $\mu = M_B$ (in Table 5), a quite stable value, $(4.75 - 4.80)$ GeV, for the Ω_{ccc} mass is found in the range of $\mu = (1.2 - 2.0)M_B$. The distinctions between the LO and NLO results on the renormalization scale can be most clearly observed in Table 2 and Fig. 4 for the Ω_{ccc} . Considering all important factors, like μ dependence and perturbative convergence, $4.75 - 4.80$ GeV is a more credible prediction for the Ω_{ccc} mass. Finally, we also evaluate the $d = 6$ GGG condensate contributions, which are found to be so small that they can be ignored in the phenomenological analysis.

Therefore, in our view, the NLO contribution is indeed important and should not be ignored. Further evaluations on the NLO corrections, e.g., to the coefficient C_{GG} of the two gluon condensation, may be needed, which will be helpful to constrain the theoretical errors for the nonperturbative contributions.

ACKNOWLEDGMENTS

We thank Chen-Yu Wang, Xiao Liu, and Xin Guan for the many useful and helpful discussions. We also thank Shi-Lin Zhu for the helpful comments.

APPENDIX A: CALCULATION OF OPERATOR RENORMALIZATION TERM

The definition of the baryon operator is shown as follows,

$$\mathcal{O}_{\Gamma_1, \Gamma_2} = \epsilon_{ijk} \left((q_1^i)^T \hat{C} \Gamma_1 q_2^j \right) \Gamma_2 q_3^k \quad (\text{A1})$$

and the next-to-leading order correction excluding the quark self-energy diagram, can be divided into two parts,

$$A = \int \frac{d^D p}{(2\pi)^D} (A_1 + A_2), \quad (\text{A2})$$

where A_1 denotes the case of the exchanging gluon between quarks q_1 and q_2 , and A_2 denotes the other cases of the exchanging gluon between quarks q_1 and q_3 and q_2 and q_3 . The corresponding amplitudes are

$$A_1 = \epsilon_{ijk} \left[i g_s \gamma_\mu \frac{-i \not{p}}{p^2} \Gamma_1 \frac{i \not{p}}{p^2} i g_s \gamma^\mu (T^a)_{ii'} (T^a)_{jj'} \right] \times [\Gamma_2 \delta_{kk'}] \frac{-i}{p^2}, \quad (\text{A3})$$

$$A_2 = \epsilon_{ijk} \left[i g_s \gamma_\mu \frac{-i \not{p}}{p^2} \Gamma_1 (T^a)_{ii'} \delta_{jj'} + \Gamma_1 \frac{-i \not{p}}{p^2} i g_s \gamma_\mu \delta_{ii'} (T^a)_{jj'} \right] \times \left[\Gamma_2 i g_s \gamma^\mu \frac{i \not{p}}{p^2} (T^a)_{kk'} \right] \frac{-i}{p^2}, \quad (\text{A4})$$

Because there are no residual infrared (IR) divergences, we only need to consider ultraviolet (UV) divergences. Therefore, the mass terms are ignored in quark propagators.

Then we can get the following form after the reduction:

$$A = -i g_s^2 \frac{1}{D} \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2)^2} B, \quad (\text{A5})$$

where

$$B = -(\gamma_\mu \gamma_\nu \Gamma_1 \gamma^\nu \gamma^\mu) (\Gamma_2) (T^a)_{ii'} (T^a)_{jj'} \delta_{kk'} \epsilon_{ijk} - D (\Gamma_1) (\Gamma_2) \left[(T^a)_{ii'} \delta_{jj'} + \delta_{ii'} (T^a)_{jj'} \right] (T^a)_{kk'} \epsilon_{ijk} - \frac{1}{2} \left([\sigma_{\mu\nu}, \Gamma_1] \right) (\Gamma_2 \sigma_{\mu\nu}) \left[(T^a)_{ii'} \delta_{jj'} + \delta_{ii'} (T^a)_{jj'} \right] (T^a)_{kk'} \epsilon_{ijk}. \quad (\text{A6})$$

According to Eq. (A5), we can get the UV divergences part of A.

$$A_{UV} = -i g_s^2 \frac{1}{4} \frac{i}{(4\pi)^2} \frac{1}{\epsilon} B \Big|_{D=4} = \frac{\alpha_s}{\epsilon} \frac{B|_{D=4}}{16\pi}, \quad (\text{A7})$$

A_{UV} is canceled by the operator renormalization term and renormalization coefficients of the quark wave function

from operator $\mathcal{O}_{\Gamma_1, \Gamma_2}$, thus, the corresponding operator renormalization term in $\overline{\text{MS}}$ scheme is

$$Z_O = 1 + \delta Z_O = 1 + \delta \left(\frac{1}{\epsilon} + \text{Log}(4\pi) - \gamma_E \right), \quad (\text{A8})$$

where

$$\delta = -\alpha_s \frac{B|_{D=4}}{16\pi} - 3 \left(-\frac{1}{2} \frac{\alpha_s}{3\pi} \right) \mathbf{I}. \quad (\text{A9})$$

where \mathbf{I} denotes the identity matrix.

At last, in this work, we can use the following relation, gotten by the Fierz transformation.

$$\epsilon_{ijk} \left((Q^i)^T \hat{C} \gamma_\lambda Q^j \right) \sigma_{\lambda\mu} Q^k = i \epsilon_{ijk} \left((Q^i)^T \hat{C} \gamma_\mu Q^j \right) Q^k. \quad (\text{A10})$$

This relation is valid in the D dimension, ($D = 4 - 2\epsilon$).

References

- [1] M. Tanabashi *et al.* (Particle Data Group), *Phys. Rev. D* **98**, 030001 (2018)
- [2] H.-X. Chen, W. Chen, X. Liu *et al.*, *Phys. Rept.* **639**, 1-121 (2016), arXiv:1601.02092
- [3] Y.-R. Liu, H.-X. Chen, W. Chen *et al.*, *Prog. Part. Nucl. Phys.* **107**, 237-320 (2019), arXiv:1903.11976
- [4] N. Brambilla, S. Eidelman, C. Hanhart *et al.*, *Phys. Rept.* **873**, 1-154 (2020), arXiv:1907.07583
- [5] R. Aaij *et al.* (LHCb), *Phys. Rev. Lett.* **119**, 112001 (2017), arXiv:1707.01621
- [6] V. A. Saleev, *Mod. Phys. Lett. A* **14**, 2615-2620 (1999), arXiv:hep-ph/9906515
- [7] M. A. Gomshi Nobary, *Phys. Lett. B* **559**, 239-244 (2003), Erratum: *Phys. Lett. B* **598**, 294-294 (2004)
- [8] M. A. Gomshi Nobary and R. Sepahvand, *Phys. Rev. D* **71**, 034024 (2005), arXiv:hep-ph/0406148
- [9] M. A. Gomshi Nobary and R. Sepahvand, *Nucl. Phys. B* **741**, 34-41 (2006), arXiv:hep-ph/0508115
- [10] Y.-Q. Chen and S.-Z. Wu, *JHEP* **08**, 144 (2011), Erratum: *JHEP* **09**, 089 (2011)
- [11] H. He, Y. Liu, and P. Zhuang, *Phys. Lett. B* **746**, 59-63 (2015), arXiv:1409.1009
- [12] S. P. Baranov and V. L. Slad, *Phys. Atom. Nucl.* **67**, 808-814 (2004), arXiv:hep-ph/0603090
- [13] J. D. Bjorken, *AIP Conf. Proc.* **132**, 390-403 (1985)
- [14] J. M. Flynn, E. Hernandez, and J. Nieves, *Phys. Rev. D* **85**, 014012 (2012), arXiv:1110.2962
- [15] W. Wang and J. Xu, *Phys. Rev. D* **97**, 093007 (2018), arXiv:1803.01476
- [16] R. A. Briceno, H.-W. Lin, and D. R. Bolton, *Phys. Rev. D* **86**, 094504 (2012), arXiv:1207.3536
- [17] S. Meinel, *Phys. Rev. D* **85**, 114510 (2012), arXiv:1202.1312
- [18] M. Padmanath, R. G. Edwards, N. Mathur *et al.*, *Phys. Rev. D* **90**, 074504 (2014), arXiv:1307.7022
- [19] Y. Namekawa *et al.* (PACS-CS), *Phys. Rev. D* **87**, 094512 (2013), arXiv:1301.4743
- [20] Z. S. Brown, W. Detmold, S. Meinel *et al.*, *Phys. Rev. D* **90**, 094507 (2014), arXiv:1409.0497
- [21] C. Alexandrou, V. Drach, K. Jansen *et al.*, *Phys. Rev. D* **90**, 074501 (2014), arXiv:1406.4310
- [22] K. U. Can, G. Erkol, M. Oka *et al.*, *Phys. Rev. D* **92**, 114515 (2015), arXiv:1508.03048
- [23] J.-R. Zhang and M.-Q. Huang, *Phys. Lett. B* **674**, 28-35 (2009), arXiv:0902.3297
- [24] T. M. Aliev, K. Azizi, and M. Savcı, *J. Phys. G* **41**, 065003 (2014), arXiv:1404.2091
- [25] Z.-G. Wang, *AAPPS Bulletin* **31**, 5 (2021)
- [26] P. Hasenfratz, R. R. Horgan, J. Kuti *et al.*, *Phys. Lett. B* **94**, 401-404 (1980)
- [27] J. Vijande, H. Garcilazo, A. Valcarce *et al.*, *Phys. Rev. D* **70**, 054022 (2004), arXiv:hep-ph/0408274
- [28] S. Migura, D. Merten, B. Metsch *et al.*, *Eur. Phys. J. A* **28**, 41 (2006), arXiv:hep-ph/0602153
- [29] W. Roberts and M. Pervin, *Int. J. Mod. Phys. A* **23**, 2817-2860 (2008), arXiv:0711.2492
- [30] A. P. Martynenko, *Phys. Lett. B* **663**, 317-321 (2008), arXiv:0708.2033
- [31] A. Bernotas and V. Simonis, *Lith. J. Phys.* **49**, 19-28 (2009), arXiv:0808.1220
- [32] J. Vijande, A. Valcarce, and H. Garcilazo, *Phys. Rev. D* **91**, 054011 (2015), arXiv:1507.03735
- [33] K. Thakkar, A. Majethiya, and P. C. Vinodkumar, *Eur. Phys. J. Plus* **131**, 339 (2016), arXiv:1609.05444
- [34] H.-X. Chen, W. Chen, X. Liu *et al.*, *Rept. Prog. Phys.* **80**, 076201 (2017), arXiv:1609.08928
- [35] X.-Z. Weng, X.-L. Chen, and W.-Z. Deng, *Phys. Rev. D* **97**, 054008 (2018), arXiv:1801.08644
- [36] G. Yang, J. Ping, P. G. Ortega *et al.*, *Chin. Phys. C* **44**, 023102 (2020), arXiv:1904.10166
- [37] M.-S. Liu, Q.-F. Lü, and X.-H. Zhong, *Phys. Rev. D* **101**, 074031 (2020), arXiv:1912.11805
- [38] M. Radin, S. Babaghodrat, and M. Monemzadeh, *Phys. Rev. D* **90**, 047701 (2014)
- [39] S.-x. Qin, C. D. Roberts, and S. M. Schmidt, *Few Body Syst.* **60**, 26 (2019), arXiv:1902.00026
- [40] P.-L. Yin, C. Chen, G. a. Krein *et al.*, *Phys. Rev. D* **100**, 034008 (2019), arXiv:1903.00160
- [41] L. X. Gutiérrez-Guerrero, A. Bashir, M. A. Bedolla *et al.*, *Phys. Rev. D* **100**, 114032 (2019), arXiv:1911.09213
- [42] K.-W. Wei, B. Chen, and X.-H. Guo, *Phys. Rev. D* **92**, 076008 (2015), arXiv:1503.05184
- [43] K.-W. Wei, B. Chen, N. Liu *et al.*, *Phys. Rev. D* **95**, 116005 (2017), arXiv:1609.02512
- [44] M. A. Shifman, A. Vainshtein, and V. I. Zakharov, *Nucl. Phys. B* **147**, 385-447 (1979)
- [45] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Nucl. Phys. B* **147**, 448-518 (1979)
- [46] L. Reinders, H. Rubinstein, and S. Yazaki, *Phys. Rept.* **127**, 1 (1985)
- [47] P. Colangelo and A. Khodjamirian, *QCD sum rules, a modern perspective*, arXiv: hep-ph/0010175
- [48] S. Narison, *Nucl. Phys. B Proc. Suppl.* **207-208**, 315-322 (2010), arXiv:1010.1959
- [49] S. Narison, *Nucl. Part. Phys. Proc.* **258-259**, 189-194 (2015), arXiv:1409.8148

- [50] R. M. Albuquerque, J. M. Dias, K. P. Khemchandani *et al.*, *J. Phys. G* **46**, 093002 (2019), arXiv:1812.08207
- [51] A. A. Ovchinnikov, A. A. Pivovarov, and L. R. Surguladze, *Int. J. Mod. Phys. A* **6**, 2025-2034 (1991)
- [52] S. Groote, J. G. Korner, and A. A. Pivovarov, *Phys. Rev. D* **78**, 034039 (2008), arXiv:0805.3590
- [53] S. Groote, J. G. Korner, and A. A. Pivovarov, *Eur. Phys. J. C* **58**, 355-382 (2008), arXiv:0807.2148
- [54] C.-Y. Wang, C. Meng, Y.-Q. Ma *et al.*, *Phys. Rev. D* **99**, 014018 (2019), arXiv:1708.04563
- [55] B. L. Ioffe, *Nucl. Phys. B* **188**, 317-341 (1981), Erratum: *Nucl. Phys. B* **191**, 591-592 (1981)
- [56] M. A. Shifman, *Quark hadron duality*, in *8th International Symposium on Heavy Flavor Physics*. World Scientific, Singapore, 7, 2000 [hep-ph/0009131]
- [57] M. Shifman, *Czech. J. Phys.* **52**, B102-B135 (2002)
- [58] S. Narison, *QCD as a Theory of Hadrons: From Partons to Confinement*, vol. 17. Cambridge University Press, 7, 2007 [hep-ph/0205006]
- [59] J. Kublbeck, M. Bohm, and A. Denner, *Comput. Phys. Commun.* **60**, 165-180 (1990)
- [60] T. Hahn, *Comput. Phys. Commun.* **140**, 418-431 (2001), arXiv:hep-ph/0012260
- [61] R. Mertig, M. Bohm, and A. Denner, *Comput. Phys. Commun.* **64**, 345-359 (1991)
- [62] V. Shtabovenko, R. Mertig, and F. Orellana, *Comput. Phys. Commun.* **207**, 432-444 (2016), arXiv:1601.01167
- [63] A. von Manteuffel and C. Studerus, *Reduze 2 - Distributed Feynman Integral Reduction*, arXiv: 1201.4330
- [64] A. V. Kotikov, *Phys. Lett. B* **254**, 158-164 (1991)
- [65] Z. Bern, L. J. Dixon, and D. A. Kosower, *Phys. Lett. B* **302**, 299-308 (1993), Erratum: *Phys. Lett. B* **318**, 649 (1993)
- [66] E. Remiddi, *Nuovo Cim. A* **110**, 1435-1452 (1997)
- [67] T. Gehrmann and E. Remiddi, *Nucl. Phys. B* **580**, 485-518 (2000), arXiv:hep-ph/9912329
- [68] X. Liu, Y.-Q. Ma, and C.-Y. Wang, *Phys. Lett. B* **779**, 353-357 (2018), arXiv:1711.09572
- [69] E. Bagan, M. Chabab, and S. Narison, *Phys. Lett. B* **306**, 350-356 (1993)
- [70] C. Dominguez, G. Gluckman, and N. Paver, *Phys. Lett. B* **333**, 184-189 (1994), arXiv:hep-ph/9406329
- [71] C. Dominguez, L. Hernandez, and K. Schilcher, *JHEP* **07**, 110 (2015), arXiv:1411.4500
- [72] S. Aoki *et al.*, *Eur. Phys. J. C* **77**, 112 (2017), arXiv:1607.00299
- [73] P. Zyla *et al.* (Particle Data Group), *PTEP* **2020**, 083C01 (2020)
- [74] R. A. Bertlmann, *Nucl. Phys. B* **204**, 387-412 (1982)
- [75] R. M. Albuquerque, S. Narison, A. Rabemananjara *et al.*, *Phys. Rev. D* **102**, 094001 (2020), arXiv:2008.01569
- [76] S. Narison, *Phys. Lett. B* **693**, 559-566 (2010), Erratum: *Phys. Lett. B* **705**, 544-544 (2011)
- [77] S. Narison, *Phys. Lett. B* **706**, 412-422 (2012), arXiv:1105.2922
- [78] S. Narison, *Phys. Lett. B* **707**, 259-263 (2012), arXiv:1105.5070