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On the microscopic shell-model version of the Bohr-Mottelson collective model

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Abstract: The recently proposed microscopic shell-model version of the Bohr-Mottelson (BM) collective model is considered in more detail in the coordinate representation. The latter possesses a clear and transparent physical meaning, which reveals several features of the new version of the collective model missed in the previous formulation. The relationship to the original BM model is considered, along with the relationships between the different limiting submodels of the microscopic version of the BM model, which closely resemble the relationships of the original Wilets-Jean and rotor models. The kinematically correct many-particle wave functions of the microscopic version of the BM model, conserving the experimentally observed integrals of motion, are shown to consist of collective irrotational-flow and intrinsic components – in accordance with the original BM unified model. The general BM Hamiltonian is obtained as a contraction limit of the microscopic many-particle nuclear Hamiltonian, or, alternatively, by restricting the latter to the scalar $O(m)$ irreducible collective space.

Keywords: shell-model version of the Bohr-Mottelson model, coordinate representation, $Sp(12, R)$ dynamical algebra, proton-neutron symplectic model

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I. INTRODUCTION

It is known that there are two fundamental models of nuclear collective motion, the Bohr-Mottelson (BM) collective model [1] and the nuclear shell model (see, for example, [2]), which have provided the central framework for the development of nuclear structure physics. The BM model was originally introduced by considering the quantization of the classical picture of surface vibrations and rotations [3]. It has demonstrated that low-lying nuclear states can be described by considering only a few macroscopic collective degrees of freedom when the intrinsic excitations lie at high energies. Conceptually, the BM model has provided the basic ideas and language with which nuclear collective motion is described. It has influenced the development of all other collective models of nuclear structure. Alternatively, the shell model includes all the many-fermion degrees of freedom and provides a general microscopic framework in terms of which the other collective models can be founded and expressed.

A natural question arises – how collective dynamics is embedded in the more complete many-fermion dynamics of the shell model? In the original formulation of these two models, this fundamental objective posed a major challenge. The solution, however, has been given through the algebraic approach by embedding the BM model in the shell model, that is, by expressing it as a submodel of the shell model (see, for example, [4, 5]). The result is the one-component $Sp(6,R)$ symplectic model [6] of nuclear collective motion, sometimes called the microscopic collective model. The embedding suggests how the collective effects can be obtained from all the single-particle fermion degrees of freedom. It has been shown that this can be achieved in an elegant way using group theory by restricting the model many-body Hamiltonian to the Hilbert state space $\mathbb{H}^{(\omega\neq(0))}$ with a definite $O(A-1)$ symmetry ω, where A is the number of protons and neutrons, or, identically, by projecting its $O(A-1)$-scalar part [7−16]. In this way, in contrast to the phenomenological collective models in which the collective (rotational and vibrational) modes are postulated, in the microscopic collective models they are derived from the Schrödinger equation for the many-particle nuclear Hamiltonian.

Recently, a fully microscopic proton-neutron symplectic model (PNSM) of nuclear collective motion with $Sp(12,R)$ dynamical algebra was introduced by considering symplectic geometry and possible collective flows in the two-component proton-neutron many-particle nuclear system [17]. The PNSM generalizes the $Sp(6,R)$ model for the case of two-component proton-neutron nuclear

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systems, which can be easily understood by embedding $Sp(6,R) \subset Sp(12,R)$. Among its dynamical symmetry limits, the PNSM contains one that has been shown to correspond to a microscopic shell-model version [18] of the BM model [1]. This correspondence has been established by considering the algebraic structures of the BM model and PNSM, which allows to consider in more detail the proper relationships of the original BM submodel limits (that is, the Wilets-Jean (WJ) [19] and rigid rotor [20, 21] models) with their microscopic shell-model counterparts within the framework of the PNSM. The correspondence of the physics shared by both the original BM model and its shell-model version has thus been demonstrated by considering the corresponding reduction chains, or, in other words, using the so called *matrix* or *algebraic representation*. In this respect, some aspects of the microscopic version of the BM model may not be sufficiently revealed in this representation.

The purpose of this study is to further consider the structure of the wave functions in the BM model and its shell-model version in the *coordinate representation*, which has a more transparent physical content. This will hopefully offer a better understanding of the intimate relationship between the original (phenomenological) and microscopic shell-model versions of the BM model and the physics behind them. This is important not only from the conceptual perspective, but also in light of the first successful applications of the new version of the BM model to the microsopic shell-model description of quadrupole dynamics in several strongly deformed [22], transitional [23], and weakly deformed [24] nuclei. The rigid- or irrotational-flow collective dynamics in the above studies were described without the use of an effective charge, which can be considered as a significant achievement of the proposed proton-neutron symplectic-based shell-model approach.

This paper is organized as follows. In Sec. II, the many-particle microscopic kinematically correct nuclear wave functions within the PNSM, which represent the microscopic shell-model analogs of the wave functions of the original BM unified (BMU) model [25, 26], are derived using the coordinate representation. In Sec. III, the general BM Hamiltonian is obtained as a contraction limit of the microscopic many-particle nuclear Hamiltonian, or, alternatively, by restricting the latter to the scalar $O(m)$ irreducible collective space $\mathbb{H}^{(\omega=(0))}$ within the framework of the symplectic based proton-neutron shell-model approach. In Sec. IV, the difference between the phenomenological and microscopic versions of the BM model, following from the consideration of the many-particle quantum mechanics of the corresponding Hilbert spaces, is given. Furthermore, the microscopic shell-model counterparts of the three well known exactly solvable limits of the phenomenological BM model are presented in Sec. V, which closely resemble the original BM submodels and their mutual relationships. In Sec. VI, the matrix representation of the many-particle wave functions of the microscopic version of the BM model is briefly considered. Finally, in the conclusion, the results are summarized.

II. COORDINATE REPRESENTATION

The use of the coordinate representation has the advantage of a more clear physical interpretation. An attempt to unify the collective and many-particle fermion degrees of freedom was conducted in the early developmental stage of the collective model by proposing the BMU model [25, 26], in which the nucleons move in a deformable shell-model potential with vibrational and rotational degrees of freedom. Thus, in the BMU model [25, 26], in the adiabatic limit, the states of a rotational band are assigned a common intrinsic state. According to this, the wave function of the nucleus in this limit can be represented in a coordinate representation as [25−27]

$$\Psi \simeq \varphi(\beta,\gamma) D^J_{KM}(\{\theta_j\}) \phi_{intr}(\xi), \tag{1}$$

where $\varphi(\beta,\gamma)$, $D^J_{KM}(\{\theta_j\})$, and $\phi_{intr}(\xi)$ denote the vibrational, rotational, and intrinsic wave functions, respectively, and ξ represents the set of fermion coordinates. Eq. (1) assumes that when the intrinsic excitations lie at high energies, the collective and intrinsic dynamics are decoupled and the low-lying nuclear states can be considered purely collective. In addition, when the rotational frequencies are considerably smaller than the vibrational frequencies, the rotational motion decouples from the vibrational degrees of freedom [1, 25], which is also reflected in (1). We notice that the BMU model approach leads to so called redundant variables (see, for example, the discussion in Sec. 11.C of Ref. [28] and the references cited therein). In what follows, we show that this problem does not arise in the microscopic shell-model approach, in which all proton-neutron degrees of freedom are properly considered.

A. Microscopic nuclear wave functions within the PNSM

In [17], it was shown that $Sp(12m,R)$ is the full dynamical group of the entire many-particle two-component proton-neutron nuclear system, spanned by all Hermitian bilinear combinations of the position $x_{is}(\alpha)$ and momentum $p_{is}(\alpha)$ many-particle relative Jacobi coordinates. The indices take the following values: $i,j = 1,2,3$, $\alpha,\beta = p,n$, and $s = 1,2,\ldots,m = A-1$. The group $Sp(12m,R)$ contains different types of motions – collective, intrinsic, cluster, etc. However, often, one restricts himself to a certain type of dominating mode in the process under consideration. Thus, by reducing $Sp(12m,R)$, one performs the separation of the $6m$ nuclear many-particle variables $\{q \equiv x_{is}(\alpha)\}$ into kinematical (internal) and dynamical (collective) variables, that is, $\{q\} = \{q_D, q_K\}$. The choice of the reduction chain depends on the concrete physical problem we want to consider. In the nuclear structure theory of collective motion, we are interested in the following reduction chain [17]:

$$Sp(12m,R) \supset Sp(12,R) \otimes O(m), \tag{2}$$

in which the first group $Sp(12,R)$ has been shown to be the group of proton-neutron collective excitations, whereas the second group $O(m)$ allows one to ensure the proper permutational symmetry of the nuclear wave functions. The latter allows us to construct kinematically correct models of nuclear motion. It is known that in formulating the nuclear many-body problem, some kinematical requirements should be satisfied by the nuclear wave function [7, 8]. First, the wave function of the nucleus should be realized microscopically, that is, it should depend on all single particle variables – spatial and spin variables. Second, the nuclear wave function should be translationally-invariant. This means that the wave function of the atomic nucleus, free from external fields, can be expressed as a product of a plane wave, describing the center-of-mass motion, and translationally-invariant wave functions, describing the internal properties of the free nucleus. The two conditions can be unified into a single one and formulated as a requirement for the wave function to be microscopically translationally-invariant. Third, the nuclear wave function should preserve the observed integrals of motion (total angular momentum, its third projection, proper permutational symmetry, *etc.*). An arbitrary wave function fulfilling the above requirements is referred to as a kinematically-correct wave function [7, 8].

In this way, the considered reduction (2) corresponds to the splitting of the microscopic many-particle configuration space $\mathbb{R}^{6m}$, spanned by the relative Jacobi vectors, into kinematical and dynamical submanifolds. It has been shown that the simplest kinematically correct wave functions within the PNSM can be classified by the quantum numbers provided by the unitary scheme chain [13, 17]:

$$\begin{array}{ccccccc} Sp(12m,R) \supset & U(6m) & \supset & U(6) & \otimes & U(m). \\ & [E_0 0\ldots 0] & & [E_1\ldots E_6] & & \left[E_1\ldots E_6\dot{0}\right] \\ & & & \cup & & \beta\cup \\ & & & G & & O(m) \\ & & & & & (\omega_1\ldots\omega_6) \\ & & & & & \delta\cup \\ & & & & & S_{m+1} \\ & & & & & [f]h \end{array} \tag{3}$$

Then, the wave function can be written in the form [17]

$$\begin{aligned}\Psi\binom{E_0 E\eta}{\beta\omega\delta[f]h}|x_1^a,\ldots,x_m^a) &\equiv \Psi\binom{E_0 E\eta}{\beta\omega\delta[f]h}|\rho^{(a_0)}, g_6^+, g_m) \\ &= \sum_{\nu^0} \Theta\binom{E_0 E\eta}{\beta\omega\nu^0}|\rho^{(a_0)}, g_6^+) D^{\omega}_{\nu^0,\delta[f]h}(g_m),\end{aligned} \tag{4}$$

where, for simplicity, the spin variables are suppressed, $\{\rho^{(a_0)}, g_6^+\}$ are the microscopic collective variables, and g_m denotes the set of complementary internal variables. The PNSM is then given a simple expression as a hydrodynamical model with wave functions comprising collective (irrotational-flow) $\Theta\binom{E_0 E\eta}{\beta\omega\nu^0}|\rho^{(a_0)}, g_6^+)$ and intrinsic (vortex) $D^{\omega}_{\nu^0,\delta[f]h}(g_m)$ components. In Eq. (4), $E = [E_1,\ldots,E_6]$ denotes both the $U(6)$ and $U(m)$ irreducible representations, and $E_0 = E_1 + \ldots + E_6$. β and δ are multiplicity indices, and η is a basis for the $U(6)$ irrep, which can be fixed by specifying the group G and its subgroups along the chain (3). From (4), we can see that the basis represented by the columns of the matrix D^{ω} is fixed by the irrep of the S_{m+1} group $[f]$ and its basis h. The latter is crucial for ensuring the proper permutational symmetry of the total nuclear wave functions. Additionally, owing to the full antisymmetry property of the nuclear wave functions, the spin content is determined by the conjugate representation $[\widetilde{f}]$. Therefore, if required, the spin part of the many-particle proton-neutron nuclear system can easily be recovered. However, for most practical applications in which only the case $S = 0$ is considered, the spin part can be dropped without lost of generality, as is done in the following.

B. Wave functions of the microscopic version of the BM model

In this study, we further consider the following reduction chain of the subgroup $U(6) \subset Sp(12,R)$ [18]:

$$\begin{array}{ccccccc} U(6) \supset & SO(6) \supset & SU_{pn}(3)\otimes SO(2) & \supset SO(3), \\ E & \upsilon & (\lambda,\mu) \leftrightarrow \nu & q \quad L \end{array} \tag{5}$$

or the equivalent to it

$$\begin{array}{l} SU(1,1)\otimes SO(6) \supset U(1)\otimes SU_{pn}(3)\otimes SO(2) \supset SO(3), \\ \lambda_\upsilon \leftrightarrow \upsilon \qquad p \qquad (\lambda,\mu) \leftrightarrow \nu \quad q \quad L \end{array} \tag{6}$$

both of which define the microscopic shell-model version of the BM model. Then, the basis index $\eta = E\upsilon\nu qL$ (or equivalently, $\eta = \lambda_\upsilon p;\upsilon\nu qL$) provides a full classification of the basis states. The nuclear wave functions along the chain (6) take the more familiar form of a direct product of radial and orbital wave functions, as we will see later. Because we consider only fully symmetric $U(6)$ and $U(m)$ irreps, the multiplicity index $\beta=1$ is dropped. Instead of the reduction chain, defined by Eqs. (3) and (5), we can use the equivalent one

$$\begin{array}{cccc} Sp(12m,R) & \supset & Sp(12,R) & \otimes & O(m) \\ & & \cup & & \cup \\ & & SU(1,1)\otimes SO(6) & & S_{m+1} \\ & & \cup & & \\ & & U(1)\otimes SU_{pn}(3)\otimes SO(2) & & \\ & & \cup & & \\ & & SO(3), & & \end{array} \tag{7}$$

obtained by combining Eqs. (2) and (6).

Recall that we can regard the configuration space of the m-quasiparticle system as a space of the real $6\times m$ matrices $\mathbb{R}^{6m}=\{x_{is}(\alpha)\equiv x_{as};\ s=1,\ldots,m;\ a=1,\ldots,6\}$. Alternatively, the quadrupole tensor of the two-component proton-neutron nuclear system can be rewritten as $Q_{ab}=\sum_{s=1}^{m}x_{as}x_{bs}\equiv Q_{ij}(\alpha,\beta)$. Then, we can view the quadrupole tensor Q as a map from the microscopic many-particle configuration space to the collective configuration space:

$$Q:\mathbb{R}^{6m}\to\mathbb{Q};\quad x\to Q(x)=\widetilde{x}x, \tag{8}$$

where $\widetilde{x}$ denotes the transpose of the matrix $x\in\mathbb{R}^{6m}$. It follows that every path $x(t)$ in $\mathbb{R}^{6m}$ has an image $Q(x(t))$ in $\mathbb{Q}$. Thus, collective motion in $\mathbb{R}^{6m}$ maps to collective motion in $\mathbb{Q}\equiv\mathbb{R}^{21}$, the latter spanned by the quadrupole moment operators $Q_{ij}(\alpha,\beta)$.

In the microscopic collective space $\mathbb{R}^{21}$ of the PNSM, there is a six-dimensional subspace spanned by $\mathbb{R}^{6}=\{Q_{ij}(p,n)=Q_{ji}(n,p)\}$, related to combined generalized quadrupole-monopole proton-neutron collective dynamics. We prove that the six components of the quadrupole tensor operators $q_{ij}=Q_{ij}(p,n)$ correspond to the microscopic collective variables $\{\rho^{(i_0)},g_3^{+}\}$. For this purpose, we perform a three-dimensional Zickendraht-Dzyublik (ZD) coordinate transformation of the many-particle variables of the proton subsystem, which in Vanagas's notation is given by [7−11, 13, 16, 29]

$$x_{is}(p)=\sum_{i_0=1}^{3}\rho_p^{(i_0)}D^{(1)_3}_{i_0,i}(g_3^{+})D^{(1)_m}_{m-3+i_0,s}(g_m), \tag{9}$$

where $\rho_p^{(i_0)}$ are three radial variables, and g_3^{+} are the standard Euler angles for the $SO(3)$ group. All six variables $\{\rho_p^{(i_0)}g_3^{+}\}$ are referred to as microscopic collective variables [7, 8, 10, 11]. Similarly, we perform the ZD transformation for the neutron subsystem many-particle coordinates $x_{is}(n)$ by replacing p with n in (9). Then, for the combined proton-neutron quadrupole operators, we obtain

$$Q_{ij}(p,n)=\sum_{i_0=1}^{3}\rho_p^{(i_0)}\rho_n^{(i_0)}D^{(1)_3}_{i_0,i}(g_3^{+})D^{(1)_3}_{i_0,j}(g_3^{+}) \tag{10}$$

Assuming $\rho^{(i_0)}=\rho_p^{(i_0)}=\rho_n^{(i_0)}$, we can see that $q_{ij}=q_{ij}(\rho^{(i_0)},g_3^{+})$ depends only on the six microscopic collective variables $\{\rho^{(i_0)},g_3^{+}\}$.

To reveal the physical meaning of the microscopic collective variables $\{\rho^{(i_0)},g_3^{+}\}$, we turn to the intrinsic body-fixed system. There is a well defined group-theoretical prescription [13] on how to obtain a given function depending on the group parameters and additional variables in the intrinsic frame of reference with respect to the group under consideration. Thus, in the intrinsic frame with respect to $SO(3)$, we obtain the well known result

$$\begin{aligned} R((g_3^{+})^{-1})q_{ij}&=\overset{\circ}{q}_{ij}\\ &=\sum_{i_0=1}^{3}(\rho^{(i_0)})^2D^{(1)_3}_{i_0,i}(e)D^{(1)_3}_{i_0,j}(e)=\rho^{(i)}\rho^{(j)}\delta_{ij}, \end{aligned} \tag{11}$$

in which the mass quadrupole tensor q_{ij} becomes diagonal in the intrinsic body-fixed coordinate system. This reveals the meaning of the radial variables $\rho^{(i_0)}$ entering (9) as principal axes values of the quadrupole mass tensor. Correspondingly, the rotational matrix $D^{(1)_3}(g_3^{+})$ takes us from the frame of reference fixed in the space to that fixed in the body.

Taking Eqs. (3) and (5) (or (6)), we can write the following for many-particle wave functions:

$$\begin{aligned} &\Psi\binom{E_0E\upsilon\nu qLM}{\omega\delta[f]h}|\mathbf{x}_1,\ldots,\mathbf{x}_m)\\ &\equiv\Psi\binom{E_0E\upsilon\nu qLM}{\omega\delta[f]h}|\rho^{(a_0)},g_6^{+},g_m)\\ &=\sum_{\nu^0}\Theta\binom{E_0E\upsilon\nu qLM}{\omega\nu^0}|\rho^{(a_0)},g_6^{+})D^{\omega}_{\nu^0,\delta[f]h}(g_m), \end{aligned} \tag{12}$$

where the $SU(1,1)$ basis states $|\lambda_\upsilon,p\rangle$ correspond to the six-dimensional harmonic oscillator basis states $|E\rangle$ [30], and the minimal Pauli allowed quantum number $E_{\min}$ corresponds to the $Sp(12,R)$ irreducible label $\langle\sigma\rangle$. As is evident from (12), the collective wave function $\Theta\binom{E_0E\upsilon\nu qLM}{\omega\nu^0}|\rho^{(a_0)},g_6^{+})$ generally depends on six radial variables $\rho^{(a_0)}$, which take the values $0\le\rho^{(a_0)}\le\infty$, and on 15 variables parameterizing the group $SO(6)$, which are denoted as $g_6^{+}=\{\alpha'_1,\alpha'_2,\ldots,\alpha'_{15}\}$. Similarly, the intrinsic wave functions depend on the $6m-21$ intrinsic coordinates g_m, which parameterize the coset space $O(m)/$

$O(m-6)$ [17].

C. Collective wave functions of the microscopic version of the BM model

Since the microscopic collective space $\mathbb{R}^6 = \{Q_{ij}(p,n) = Q_{ji}(n,p)\}$ is six-dimensional, 15 of the 21 microscopic collective parameters $\{\rho^{(a_0)}, g_6^+\}$ should be zero. For the non-zero collective variables, we choose $\rho^{(a_0)} \equiv \rho^{(i_0)} = \rho_p^{(i_0)} = \rho_n^{(i_0)}$ for $a_0 \equiv i_0 = 1,2,3$ and $\rho^{(a_0)} = 0$ for $a_0 = 4,5,6$, as well as $g_6^+ = \{\alpha_1' \neq 0, \alpha_2' \neq 0, \alpha_3' \neq 0, \alpha_4' = 0, \ldots, \alpha_{15}' = 0\} \equiv \{g_3^+, 0, \ldots, 0\}$. The set of three parameters $g_3^+ = \{\alpha_1', \alpha_2', \alpha_3'\}$ denotes the usual Euler angles. Then, for the microscopic collective wave functions, we can write

$$\begin{aligned}&\Theta\binom{E_0 E\upsilon\nu qLM}{\omega\nu^0}|\rho^{(i_0)}, g_3^+)\\ &= \sum_K \Theta\binom{E_0 E\upsilon\nu qLK}{\omega\nu^0}|\rho^{(i_0)}) D_{KM}^L(g_3^+),\end{aligned} \tag{13}$$

where $\Theta\binom{E_0 E\upsilon\nu qLK}{\omega\nu^0}|\rho^{(i_0)})$ are obtained using the initial wave functions $\Psi\binom{E_0 E\upsilon\nu qLM}{\omega\delta[f]h}|\mathbf{x}_1, \ldots, \mathbf{x}_m)$ via the following expression [7−11]:

$$\begin{aligned}&\Theta\binom{E_0 E\upsilon\nu qLK}{\omega\nu^0}|\rho^{(i_0)})\\ &= \Psi\binom{E_0 E\upsilon\nu qLM}{\omega\delta[f]h}|\mathbf{x}_1 = 0, \ldots, \mathbf{x}_{m-2} = \rho^{(1)}\delta_{1,m-2},\\ &\quad \mathbf{x}_{m-1} = \rho^{(2)}\delta_{2,m-1}, \mathbf{x}_m = \rho^{(3)}\delta_{3,m}).\end{aligned} \tag{14}$$

Instead of the three radial variables $\{\rho^{(i_0)}\}$, we can use the three variables $\{r, \widetilde{\theta}_1, \widetilde{\theta}_2\}$, defined by the transformation [7, 8, 13]

$$\begin{aligned}\rho^{(1)} &= r\sin\widetilde{\theta}_1\sin\widetilde{\theta}_2,\\ \rho^{(2)} &= -r\sin\widetilde{\theta}_1\cos\widetilde{\theta}_2,\\ \rho^{(3)} &= r\cos\widetilde{\theta}_1,\end{aligned} \tag{15}$$

where $r = \sqrt{\sum_s (x_s^2(p) + x_s^2(n))}$ is the global radius. Then, instead of the microscopic collective variables $\{\rho^{(i_0)}, g_3^+\}$ (or $\{q_{ij}\}$), we can use the equivalent set of six collective variables $\{r, \widetilde{\theta}_1, \widetilde{\theta}_2, g_3^+\}$, where the angles $\{\widetilde{\theta}_1, \widetilde{\theta}_2, g_3^+\}$ define a point on the five-sphere S_5. Using these, the collective wave functions of the microscopic version of the BM model can alternatively be written as

$$\begin{aligned}&\Theta\binom{E_0 E\upsilon\nu qLM}{\omega\nu^0}|r, \widetilde{\theta}_1, \widetilde{\theta}_2, g_3^+)\\ &= R_{E\upsilon}(r) D^{\upsilon}_{0,\nu qLM\omega\nu^0}(\widetilde{\theta}_1, \widetilde{\theta}_2, g_3^+),\end{aligned} \tag{16}$$

where $R_{E\upsilon}(r)$ is the radial wave function of the six-dimensional oscillator, and $\overline{0}$ represents the scalar $SO(6)$ irrep because r is an $SO(6)$ invariant variable. Introducing the notation $Y^{\upsilon}_{\nu qLM}(\Omega_5) = \sqrt{d_\upsilon} D^{\upsilon}_{0,\nu qLM\omega\nu^0}(\widetilde{\theta}_1, \widetilde{\theta}_2, g_3^+)$, we obtain the coordinate representation of the microscopic collective wave functions, classified by Eqs. (5) (or (6)) and introduced (up to a numerical factor) in [18],

$$\Theta\binom{E_0 E\upsilon\nu qL}{\omega\nu^0}|r, \widetilde{\theta}_1, \widetilde{\theta}_2, g_3^+) = (d_\upsilon)^{-1/2} R_{E\upsilon}(r) Y^{\upsilon}_{\nu qLM}(\Omega_5) \tag{17}$$

in terms of the $SO(6)$ Dragt [31, 32] spherical harmonics $Y^{\upsilon}_{\nu qLM}(\Omega_5)$ depending on the five angles $\Omega_5 = \{\widetilde{\theta}_1, \widetilde{\theta}_2, g_3^+\}$. Note that, generally, other choices for the parameterization of the five-sphere exist.

D. Relation to the BM wave functions

Now, we return to the full many-particle nuclear wave functions written in the form

$$\begin{aligned}&\Psi\binom{E_0 E\upsilon\nu qLM}{\omega\delta[f]h}|x_1^a, \ldots, x_m^a)\\ &\equiv \Psi\binom{E_0 E\upsilon\nu qLM}{\omega\delta[f]h}|\rho^{(i_0)}, g_3^+, g_m)\\ &= \sum_{\nu^0} \Theta\binom{E_0 E\upsilon\nu qLM}{\omega\nu^0}|\rho^{(i_0)}, g_3^+) D^{\omega}_{\nu^0,\delta[f]h}(g_m),\end{aligned} \tag{18}$$

where the collective wave functions $\Theta\binom{E_0 E\upsilon\nu qL}{\beta\omega\nu^0}|\rho^{(i_0)}, g_3^+)$ allow us to establish a relationship with the original BM collective wave functions depending on the phenomenological parameters (β, γ). For this purpose, we consider the following transformation [7, 8, 13]:

$$\begin{aligned}r^2 &= (\rho^{(1)})^2 + (\rho^{(2)})^2 + (\rho^{(3)})^2,\\ \beta\sin\gamma &= \frac{1}{\sqrt{2}}\left((\rho^{(1)})^2 - (\rho^{(2)})^2\right),\\ \beta\cos\gamma &= \frac{1}{\sqrt{6}}\left((\rho^{(1)})^2 + (\rho^{(2)})^2 - 2(\rho^{(3)})^2\right),\end{aligned} \tag{19}$$

where r is again the global radius. The collective wave functions thus become

$$\begin{aligned}&\Theta\binom{E_0 E\upsilon\nu qLM}{\omega\nu^0}|\rho^{(i_0)}, g_3^+)\\ &\equiv \Theta\binom{E_0 E\upsilon\nu qLM}{\omega\nu^0}|r, \beta, \gamma, g_3^+)\\ &= R_{E\upsilon}(r)\Theta\binom{E_0 E\upsilon\nu qLM}{\omega\nu^0}|\beta, \gamma, g_3^+),\end{aligned} \tag{20}$$

where $\Theta\binom{E_0 E\upsilon\nu qLM}{\omega\nu^0}|\beta, \gamma, g_3^+)$ gives the microscopic shell-model counterpart of the BM collective functions. The dependence on the Euler angles can be further taken explicitly using the following expression:

$$\begin{aligned}&\Theta\left(\begin{smallmatrix}E_0 E\upsilon\nu qLM\\ \omega\nu^0\end{smallmatrix}\Big|\beta,\gamma,g_3^+\right)\\&=\sum_K \Theta\left(\begin{smallmatrix}E_0 E\upsilon\nu qLK\\ \omega\nu^0\end{smallmatrix}\Big|\beta,\gamma\right)D^L_{KM}(g_3^+),\end{aligned}\tag{21}$$

where $\Theta\left(\begin{smallmatrix}E_0 E\upsilon\nu qLK\\ \omega\nu^0\end{smallmatrix}\big|\beta,\gamma\right)$ and $D^L_{KM}(g_3^+)$ represent the microscopic shell-model counterparts of the vibrational and rotational collective wave functions of Eq. (1). Then, the full-shell model analog of the BMU model wave functions of the nucleus becomes

$$\begin{aligned}&\Psi\left(\begin{smallmatrix}E_0 E\upsilon\nu qLM\\ \omega\delta[f]h\end{smallmatrix}\Big|x_1^a,\ldots,x_m^a\right)\\&\equiv\Psi\left(\begin{smallmatrix}E_0 E\upsilon\nu qLM\\ \omega\delta[f]h\end{smallmatrix}\Big|r,\beta,\gamma,g_3^+,g_m\right)\\&=\sum_{\nu^0,K}R_{E\upsilon}(r)\Theta\left(\begin{smallmatrix}E_0 E\upsilon\nu qLK\\ \omega\nu^0\end{smallmatrix}\Big|\beta,\gamma\right)D^L_{KM}(g_3^+)D^{\omega}_{\nu^0,\delta[f]h}(g_m),\end{aligned}\tag{22}$$

where, in addition, the radial wave function $R_{E\upsilon}(r)$, depending on the global radius r, appears in the decomposition. Here, an important difference appears between the phenomenological and microscopic shell-model approaches. From Eq. (22), it follows that owing to the sum over ν^0, in the microscopic version of the BM model, the collective and intrinsic dynamics are strongly coupled, in contrast with the phenomenological case (cf. Eq. (1)). Decoupling will appear only for the one-dimensional scalar $O(m)$ irreducible representation $\omega=(0)$, relevant only to doubly closed shell nuclei, in which case the sum over ν^0 disappears. Hence, the microscopic wave functions corresponding to the original ones of the BMU model are obtained from Eq. (22) only for the scalar $O(m)$ irreducible representation $\omega=(0)$ (or, equivalent to it, the scalar symplectic $Sp(12,R)$ irrep $\langle\sigma\rangle=(0)$). As shown later, this leads to some specific artificial features of the collective dynamics in many-particle two-component proton-neutron nuclear systems. However, we first consider the derivation of the original BM collective Hamiltonian.

III. DERIVATION OF THE BOHR-MOTTELSON HAMILTONIAN

In this section, we show how the standard BM Hamiltonian can be derived from the microscopic many-particle nuclear Hamiltonian. First, we obtain the Bohr Hamiltonian, which is just the kinetic energy term. To obtain the kinetic energy operator, we must consider the momentum observables. In the BM model, they are canonically conjugate to the position coordinates $\{\alpha_\nu\}$, that is,

$$\pi^\nu=-\mathrm{i}\hbar\frac{\partial}{\partial\alpha_\nu}\,,\qquad \nu=0,\pm1,\pm2.\tag{23}$$

The BM model variables $\{\alpha_\nu,\pi^\mu\}$ thus obey the standard Heisenberg-Weyl commutation relations

$$[\alpha_\mu,\pi^\nu]=\mathrm{i}\hbar\delta^\nu_\mu.\tag{24}$$

In microscopic nuclear theory, the phenomenological surface parameters $\{\alpha_\nu\}$ are replaced by the quadrupole moment operators Q_ν [4, 5, 18], that is, $\{\alpha_\nu\}\to Q_\nu$, which have a well-defined expression in the many-particle coordinates. Furthermore, we include the monopole degree of freedom and consider the mapping $\{\alpha_{\lambda\mu}\}\to\{q_{ij}=Q_{ij}(p,n)\}$ [18], where $\lambda=0,2$ and $\mu=-\lambda,\ldots,\lambda$. To obtain the Bohr (kinetic energy) Hamiltonian, we use the Laplace operator in the microscopic collective variables $q_{ij}=q_{ij}(\rho^{(i_0)},g_3^+)$ [10, 11]:

$$\begin{aligned}\nabla^2_{\rm phen}=&\sum_{i\geq j=1}^{3}\frac{\partial^2}{\partial(q_{ij})^2}=\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\\&+2\left(\frac{1}{X-Y}+\frac{1}{X-Z}\right)\frac{\partial}{\partial X}+2\left(\frac{1}{Y-Z}+\frac{1}{Y-X}\right)\frac{\partial}{\partial Y}\\&+2\left(\frac{1}{Z-X}+\frac{1}{Z-Y}\right)\frac{\partial}{\partial Z}+\frac{2}{(X-Y)^2}\overline{L}^2_{12}\\&+\frac{2}{(X-Z)^2}\overline{L}^2_{13}+\frac{2}{(Y-Z)^2}\overline{L}^2_{23},\end{aligned}\tag{25}$$

in which the notations $(\rho^{(1)})^2\equiv X$, $(\rho^{(2)})^2\equiv Y$, and $(\rho^{(1)})^2\equiv Z$ are used. To give Eq. (25) a more familiar form, we express the components of the quadrupole tensor in $SO(3)$ irreducible terms,

$$q_{lm}=\sum_{i\geq j=1}^{3}C^{1\ 1\ l}_{i\ j\ m}q_{ij}=\sum_{m'}p^l_{m'}D^l_{m'm}(g_3^+),\tag{26}$$

where $l=2,0$, and $p^l_m=\sum_{i_0}(\rho^{(i_0)})^2C^{1\ \ 1\ \ l}_{i_0\ i_0\ m}$. In particular, for $l=0$, we get

$$\sqrt{3}p^0_0=X+Y+Z=r^2\equiv p_0.\tag{27}$$

For $l=2$, using the Clebsch-Gordan coefficients in the Cartesian basis, we find that p^2_m is nonzero only for two values of m. These values are denoted as 11 and 22, and we explicitly express p^2_{11} and p^2_{22} as

$$\begin{aligned}p^2_{11}&=\frac{1}{\sqrt{6}}(X+Y-2Z)\\p^2_{22}&=\frac{1}{\sqrt{2}}(X-Y),\end{aligned}\tag{28}$$

which coincide with the last two equations of (19).

In the variables (19), the operator (25) takes the following form [10, 11]:

$$\nabla^2_{\rm phen} = 3\frac{\partial^2}{\partial (r^2)^2} + 4\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \sum_{i>j=1}^{3}\frac{\overline{L}^2_{ji}(g_3^+)}{\mathfrak{I}_{ji}(\beta\gamma)}\right]. \quad (29)$$

where

$$\mathfrak{I}_{12} = 4\beta^2 \sin^2\gamma,$$
$$\mathfrak{I}_{13} = 4\beta^2 \sin^2\left(\gamma - \frac{2\pi}{3}\right),$$
$$\mathfrak{I}_{23} = 4\beta^2 \sin^2\left(\gamma - \frac{4\pi}{3}\right). \quad (30)$$

We find that the second term in Eq. (29) coincides with the Laplace operator of the phenomenological rotation-vibration BM model,

$$\nabla^2_{\rm Bohr} = \left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2 \sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \sum_{i>j=1}^{3}\frac{\overline{L}^2_{ji}(g_3^+)}{\mathfrak{I}_{ji}(\beta\gamma)}\right] \quad (31)$$

and, in addition, Eq. (29) contains the kinetic energy term related to the radial (monopole) oscillations. The full BM Hamiltonian is then obtained by adding the potential energy term, that is,

$$H_{\rm BM} = -\frac{\hbar^2}{2\mathfrak{B}}\nabla^2_{\rm Bohr} + V(\beta,\gamma). \quad (32)$$

Since $V(\beta,\gamma)$ can be expressed in terms of the microscopic quadrupole moment operators q_{ij} because $[q\times q]^{(0)} \sim \beta^2$ and $[q\times q\times q]^{(0)} \sim \beta^3\cos 3\gamma$, any BM Hamiltonian of the form (32) immediately defines a microscopic shell-model Hamiltonian

$$H = K(p,n) + V(q), \quad (33)$$

where the operator $-\frac{\hbar^2}{2\mathfrak{B}}\nabla^2_{\rm Bohr}$ is replaced by the many-particle kinetic energy $K(p,n) = \frac{1}{2M}\sum_{is} p_{is}(p)p_{is}(n)$. $V(q)$ is a rotationally invariant function that can be built up from different powers of the quadrupole moment operators q_{ij}.

It is convenient, however, to keep all six collective variables $\{\rho^{(i_0)}, g_3^+\}$ or those equivalent to them $\{q_{ij}\}$ and use $\nabla^2_{\rm phen}$ (29). Including the radial degrees of freedom, the microscopic analog of the generalized quadrupole-monopole BM Hamiltonian can be written in the form

$$H(q_{ij}) = -\frac{\hbar^2}{2M}\nabla^2_{\rm phen} + V(r,\beta,\cos 3\gamma). \quad (34)$$

This Hamiltonian acts in the phenomenological Hilbert space $\mathbb{H}_{\rm phen}$, spanned by the complete set of wave functions depending on the six collective variables $\{r,\beta,\gamma,g_3^+\}$ or those equivalent to them $\{q_{ij}\}$.

An important property used in the derivation of the phenomenological BM Hamiltonian from the microscopic many-particle nuclear Hamiltonian is the exploitation of the Hermitian conjugate pair of the position (q_{ij}) and momentum $(p_{ij} = -i\hbar\partial/\partial q_{ij})$ collective variables. However, using the microscopic collective variables $Q_{ij}(p,n)$, it can easily be checked that

$$p_{ij} \equiv P_{ij}(p,n) = M\dot{Q}_{ij}(p,n) = \sum_{s=1}^{A}\Big(p_{is}(p)x_{js}(n) + x_{is}(p)p_{js}(n)\Big) \neq -i\hbar\frac{\partial}{\partial q_{ij}}. \quad (35)$$

From Eq. (35), it becomes clear that the quantization of the BM model (cf. Eqs. (23) and (24)) is not correct. New commutation relations emerge,

$$[\hat{q}_{ij}, \hat{p}_{kl}] = i\hbar\left(\delta_{il}\hat{q}_{jk} + \delta_{ik}\hat{q}_{jl} + \delta_{jl}\hat{q}_{ik} + \delta_{jk}\hat{q}_{il}\right), \quad (36)$$

which generate the Lie algebra of a general collective motion group in three dimensions: $GCM(3) = \{L_{ij}, q_{ij}, p_{ij}\}$, where $L_{ij} = \sum_{\alpha,s}(x_{si}(\alpha)p_{sj}(\alpha) - x_{sj}(\alpha)p_{si}(\alpha))$. The $GCM(3)$ model is a slightly extended version of the original $CM(3)$ model of Weaver, Biedenharn, and Cusson [33−35] (see also [36]), which also includes monopole degrees of freedom. A characteristic feature of the new spectrum generating algebra of the $GCM(3)$ model is that it has irreps with different intrinsic angular momenta (vorticities).

The original BM commutation relations, however, can be obtained as a contraction limit from those of the $GCM(3)$ model. Following the procedure of Inonu and Wigner [37], the monopole and quadrupole moments of the nucleus can be expressed in units of a small parameter ε, which is assigned a value $4\varepsilon = 1/\langle q_0\rangle$, where $\langle q_0\rangle$ is the mean value of the monopole moment of the nucleus in its low-energy states. In these units, the monopole/quadrupole observables are given by [38]

$$Q_0 = \varepsilon q_0 = \varepsilon(q_{11} + q_{22} + q_{33}), \quad (37)$$

$$P^0 = \varepsilon p^0 = \varepsilon(p_{11}+p_{22}+p_{33}), \tag{38}$$

$$Q_{2\nu} = \varepsilon q_{2\nu}, \quad P^{2\nu} = \varepsilon(-1)^\nu p_{2,-\nu}. \tag{39}$$

With the commutation relations of Eq. (36), it is easily determined that

$$[Q_0, P^0] = \mathrm{i}\hbar I + O(\varepsilon^2), \quad [Q_{2\mu}, P^{2\nu}] = \mathrm{i}\hbar\delta^\nu_\mu + O(\varepsilon), \tag{40}$$

and that the commutators $[Q_0, P^{2\nu}]$, $[Q_{2\nu}, P^0]$, $[Q_{2\mu}, Q_{2\nu}]$, and $[P^{2\mu}, P^{2\nu}]$ are $O(\varepsilon)$. These are microscopic versions of the Heisenberg-Weyl commutation relations assumed in the BM model. It follows then that the BM collective Hamiltonian is also obtained in the considered contraction limit from the microscopic many-particle nuclear Hamiltonian.

The same result (29) can also be directly obtained using the expression for the many-particle kinetic-energy (that is, the $3m$ Laplace operator) in terms of the ZD variables ρ^{i_0}, g_3^+, and g_m and their derivatives [9, 11, 13]:

$$\begin{aligned}
\Delta \equiv \nabla^2 &= \sum_{is}\frac{\partial}{\partial(x_{is}(p))}\frac{\partial}{\partial(x_{is}(n))} \\
&= \sum_{i'_0}\frac{\partial^2}{\partial(\rho^{(i'_0)})^2} + |m-3|\sum_{i'_0}\frac{1}{\rho^{(i'_0)}}\frac{\partial}{\partial(\rho^{i'_0})} \\
&\quad + 2\sum_{i'_0<i''_0}\frac{1}{(\rho^{(i'_0)})^2-(\rho^{(i''_0)})^2}\left[\rho^{(i'_0)}\frac{\partial}{\partial\rho^{(i'_0)}} - \rho^{(i''_0)}\frac{\partial}{\partial\rho^{(i''_0)}}\right] \\
&\quad - \sum_{i'}\sum_{s'=1}^{m-3}\frac{J^2_{s',m-3+i'}}{(\rho^{(i')})^2} \\
&\quad - \sum_{i'_0<i''_0}\frac{(\rho^{(i'_0)})^2+(\rho^{(i''_0)})^2}{((\rho^{(i'_0)})^2-(\rho^{(i''_0)})^2)^2}\left[J^2_{m-3+i'_0,m-3+i''_0} + \mathcal{L}^2_{i'_0 i''_0}\right] \\
&\quad - 4\sum_{i'_0<i''_0}\frac{(\rho^{(i'_0)})\rho^{(i''_0)})}{((\rho^{(i'_0)})^2-(\rho^{(i''_0)})^2)^2}J_{m-3+i'_0,m-3+i''_0}\mathcal{L}_{i'_0 i''_0},
\end{aligned} \tag{41}$$

assuming again that $\rho^{(i_0)} = \rho_p^{(i_0)} = \rho_n^{(i_0)}$. In (41), $\mathbf{J}$ and $\mathcal{L}$ are the infinitesimal angular momentum operators of the groups $O(m)$ and $SO(3)$ and are defined in the intrinsic frame, with respect to the $O(m)$ and $SO(3)$ groups, respectively. Similar expressions for the many-particle kinetic energy have been obtained in Refs. [5, 14, 39–46]. The operators $\mathbf{J}$ of intrinsic rotations were first introduced in [47] and called quasi-momentum operators, later to be rediscovered by other authors [35, 48, 49] (see also [4, 5, 38]) and referred to as vortex-spin operators. From Eq. (41), we see that the many-particle kinetic energy depends on the number of particles and the $O(m)$ generators. This dependence disappears if we restrict the many-particle Laplace operator (41) to the $O(m)$-scalar space, corresponding to the original irrotational-flow BM model, in which the many-particle kinetic energy reduces to

$$\begin{aligned}
\Delta^{(0)} &= \sum_{i'_0}\frac{\partial^2}{\partial(\rho^{(i'_0)})^2} \\
&\quad + 2\sum_{i'_0<i''_0}\frac{1}{(\rho^{(i'_0)})^2-(\rho^{(i''_0)})^2}\left[\rho^{(i'_0)}\frac{\partial}{\partial\rho^{(i'_0)}} - \rho^{(i''_0)}\frac{\partial}{\partial\rho^{(i''_0)}}\right] \\
&\quad - \sum_{i'_0<i''_0}\frac{(\rho^{(i'_0)})^2+(\rho^{(i''_0)})^2}{((\rho^{(i'_0)})^2-(\rho^{(i''_0)})^2)^2}\mathcal{L}^2_{i'_0 i''_0}.
\end{aligned} \tag{42}$$

This expression is the same as Eq. (25); therefore, using (19), we can obtain the final result (29) with the useful identifications for the $SO(3)$ angular momentum operators $\overline{L}_{ij} = \mathcal{L}_{ij}$ in the intrinsic frame and irrotational-flow moments of inertia $\mathfrak{I}_{ij} = \dfrac{((\rho^{(i)})^2-(\rho^{(j)})^2)^2}{(\rho^{(i)})^2+(\rho^{(j)})^2}$.

From Eq. (41), we find that, in the case of the general non-scalar $O(m)$ irrep $\omega \neq (0)$, the many-particle kinetic-energy operator couples the collective to the intrinsic dynamics, and the latter is associated with the rotations in many-particle index space. Decoupling of the collective degrees of freedom from internal dynamics is achieved only for the scalar $O(m)$ irrep $\omega = (0)$, corresponding to the case of doubly-closed shell nuclei, which exhibit irrotational-flow dynamics of the BM type. We also observe that the true many-particle kinetic energy operator possesses a considerably richer structure than the original Bohr Hamiltonian. This implies that in the practical shell-model calculations within the microscopic version of the BM model, we use the full many-particle kinetic energy instead of the simpler Bohr Hamiltonian.

IV. PHENOMENOLOGICAL AND MICROSCOPIC COLLECTIVE HILBERT SPACES

The collective wave functions of the microscopic version of the BM model are defined by Eq. (20). They span the microscopic collective subspace $L^2(\mathbb{R}^6)$ of the many-particle Hilbert space $\mathbb{H}$. Then, the microscopic shell-model counterpart of the BM collective functions $\Theta\binom{E_0 E v \nu q LM}{\omega\nu^0}|\beta,\gamma,g_3^+)$ spans $L^2(\mathbb{R}^5) \subset L^2(\mathbb{R}^6)$. The Hilbert space is characterized by all exact integrals of motion observed experimentally and denoted collectively as $\Lambda_0 = \{\pi, L, M, [f], h, A, T_3 = 1/2(Z-N)\}$. The last two integrals of motion indicate that the numbers of protons and neutrons, constituting the nucleus, are preserved. Additionally, a new integral of motion appears – the $O(m)$ irreducible representation ω– which is related to the collective effects. The full collective Hilbert space is then a direct sum of $O(m)$ irreducible collective subspaces: $\mathbb{H}^{(\Lambda_0)} = \mathbb{H}^{(\Lambda_0\omega\Gamma)} \oplus \mathbb{H}^{(\Lambda_0\omega'\Gamma')} \oplus \ldots$, where Γ denotes the set of remaining quantum numbers required to classify the nuclear states and is determined by the reduction chains (3) and (5) (or equivalently (7)).

The phenomenological collective space of the BM model corresponds to $\omega=(0)$ [7, 8, 10, 11], that is, $\mathbb{H}_{\rm phen}=\mathbb{H}^{(\Lambda_0\omega=(0)\Gamma)}$. A specific feature of this violated permutational symmetry space $\mathbb{H}^{(\Lambda_0\omega=(0)\Gamma)}$ is that it gives a "freezing" of the intrinsic collective structure of the used Hamiltonians and makes them similar to those in BM theory, associated only with the irrotational-flow collective dynamics. Then, only the high-energy excitations will appear, which are related to the giant resonance degrees of freedom. However, for $\omega=(0)$, from the Young scheme $[f]=[m]$ of S_{m+1} it follows $[\widetilde{f}]=[1^m]$ for the conjugate spin symmetry, which is impossible for $m>4$ due to the Pauli principle. This means that the Pauli allowed Hilbert subspaces of the microscopic version of the BM model are spanned by the $O(m)$ irreducible subspaces $\mathbb{H}^{(\Lambda_0\omega\Gamma)}$ with $\omega\geq\omega_{\rm min}$, where $\omega_{\rm min}$ does not permit $O(m)$-scalar values for $m>4$. Thus, strictly speaking, the original BM collective model is not kinematically correct. This property is lost when $\omega=(0)$, which means that in phenomenological models, the effects related to the multidimensional particle-index space $O(m)$ are ignored, immediately resulting in the violation of the Pauli principle.

Therefore, it is clear that to restore the correct permutational symmetry and recover the full proton-neutron quadrupole-monopole collective dynamics, we must consider the Pauli allowed subspaces $\mathbb{H}^{(\Lambda_0\omega\neq(0)\Gamma)}$ of the full many-particle Hilbert space. In other words, from the hydrodynamical content of the PNSM [17], we know that for the more complete description of nuclear many-particle dynamics, we must include the intrinsic (vortex) degrees of freedom. The intrinsic (vortex) subdynamics in the PNSM is represented by an intrinsic $U(6)$ structure (or an $Sp(12,R)$ symplectic bandhead $\langle\sigma\rangle\equiv\omega$), which in shell-model terms is associated with the valence-shell proton-neutron degrees of freedom. It turns out that the presence of this intrinsic structure significantly modifies the proton-neutron rigid-flow quadrupole dynamics, particularly that in the rotor model and WJ limits of the microscopic version of the BM model.

V. LIMITING CASES OF THE MICROSCOPIC VERSION OF THE BM MODEL

The Laplace collective operator in the microscopic version of the BM model is given by

$$\nabla^2=\frac{1}{r^5}\frac{\partial}{\partial r}r^5\frac{\partial}{\partial r}-\frac{\Lambda^2}{r^2},\tag{43}$$

where Λ^2 is the $SO(6)$ Casimir operator. Its concrete form depends on the parameterization of the five-sphere S_5. For example, using the Zickendraht parameterization [50],

$$\begin{aligned}r_p&=\frac{r}{\sqrt{2}}\sqrt{1-\sin\alpha\sin\phi},\\ r_n&=\frac{r}{\sqrt{2}}\sqrt{1+\sin\alpha\sin\phi},\end{aligned}\tag{44}$$

consisting of the three Euler angles $g_3^+=(\theta_1,\theta_2,\theta_3)$, the hyper-radius r and two internal angles (α,ϕ). The Euler angles define the intrinsic body-fixed system with respect to the laboratory system, whereas the variables (r,α,ϕ) define the shape of the nucleus. The five-sphere is thus defined by the five angles $\Omega_5=\{\alpha,\phi,g_3^+\}$. Then, the $SO(6)$ Casimir operator is expressed in terms of the Zickendraht coordinates in the form [50]

$$\begin{aligned}\Lambda^2=&4\left\{\frac{\partial^2}{\partial\alpha^2}+2\cot2\alpha\frac{\partial}{\partial\alpha}+\frac{1}{\sin^2\alpha}\frac{\partial^2}{\partial\phi^2}-\frac{\cos\alpha}{\sin^2\alpha}\overline{L}_z i\frac{\partial}{\partial\phi}\right\}\\&-\left[A(\alpha)\overline{L}_x^2+B(\alpha)\overline{L}_y^2+C(\alpha)\overline{L}_z^2\right],\end{aligned}\tag{45}$$

where

$$\begin{aligned}&A(\alpha)=\frac{1}{(1+\sin\alpha)},\quad B(\alpha)=\frac{1}{(1-\sin\alpha)},\\&C(\alpha)=\frac{1}{2\sin^2\alpha}.\end{aligned}\tag{46}$$

and $\{\overline{L}_k\}$ are the components of the angular momentum operator in the intrinsic principal axis system. Note that the fourth term in the curly brackets and the terms in square brackets couple rotational dynamics with the shape vibrations of the nuclear surface.

The interaction in this parameterization will depend on the three internal coordinates (r,α,ϕ), and the Hamiltonian of the microscopic version of the BM model can then be written in the form

$$H=-\frac{\hbar^2}{2M}\nabla^2+V(r,\alpha,\phi),\tag{47}$$

where the Laplacian is given by Eqs. (43) and (45). The six-dimensional spherical harmonics, which are eigenfunctions of Λ^2, can thus be represented in the form [50]

$$Y^{\upsilon}_{\nu qLM}(\alpha,\phi,g_3^+)={\rm e}^{{\rm i}\nu\phi}\sum_{K=-L}^{L}f^{\upsilon}_{\nu qLK}(\alpha)D^L_{KM}(g_3^+),\tag{48}$$

where K takes only even (odd) values for even (odd) υ. For the γ-unstable WJ model limit of the microscopic version of the BM model, corresponding to the asymmetric rotor, and for the more general case of an arbitrary potential $V(r,\alpha,\phi)$, we can replace $D^L_{KM}(g_3^+)$ in Eq. (48) with the symmetrized rotational function $|LKM\rangle_{D_2} = \frac{1}{\sqrt{2(1+\delta_{K0})}}[D^L_{KM}(g^+_3)+(-1)^L D^L_{-KM}(g^+_3)]$, which accounts for the D_2 symmetry of the non-axial rotor.

A. Harmonic vibrator submodel

The Hamiltonian for the spherical vibrator submodel of the microscopic version of the BM model is defined by the Hamiltonian of the six-dimensional harmonic oscillator

$$H_{HV} = -\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}m\omega^2 r^2, \tag{49}$$

whose equidistant energies are given by

$$\mathcal{E}(E) = \left(E+\frac{6}{2}\right)\hbar\omega, \tag{50}$$

with $E = 0,1,2,\ldots$ [18]. Strictly speaking, this situation is valid only for doubly closed shell nuclei, or for the scalar $\langle\sigma\rangle = (0)$ irreducible representation of the $Sp(12,R)$ dynamical group. For open-shell nuclei, owing to the Pauli principle, the number of oscillator quanta E starts from a minimal Pauli allowed value $E_{\min} = \sigma_1 + \ldots + \sigma_6$, which as explicitly shown, is determined by the symplectic bandhead structure $\langle\sigma\rangle \neq (0)$.

Note that there is also an approximate solution for the six-dimensional displaced (or deformed) oscillator with a potential

$$V(r) = \frac{1}{2}C(r-r_0)^2, \tag{51}$$

which is valid in the limit of small oscillations around the equilibrium value r_0. This is given in terms of two quantum numbers (n,υ) [51],

$$\mathcal{E}(n,\upsilon) \cong \epsilon\left(n+\frac{1}{2}\right) + \frac{1}{2Mr_0^2}\left[\upsilon(\upsilon+4)+\frac{15}{4}\right], \tag{52}$$

where $\epsilon = \sqrt{C/M}$. The first term gives a harmonic vibrational spectrum with $n = 0,1,2,\ldots$, and the second term gives a quasirotational spectrum with $\upsilon = 0,1,2,\ldots$.

B. Wilets-Jean submodel

A six-dimensional analog of the WJ limit of the microscopic version of the BM model is obtained for $V = V(r)$. This limit is thus invariant under the $SO(6)$ transformations. The microscopic analog of the WJ Hamiltonian is an $SO(6)$ invariant, and its eigenvectors occur in multiplets that span irreducible representations of the $SO(6)$ group. The energies are labeled by the $SO(6)$ quantum number υ.

The energies and radial wave functions can be found as solutions of the eigenvalue equation [18]

$$\left[-\frac{\hbar^2}{2M}\left(\nabla^2 - \frac{\upsilon(\upsilon+4)}{r^2}\right) + V(r)\right]R_{E\upsilon}(r) = \mathcal{E}_{E\upsilon}R_{E\upsilon}(r), \tag{53}$$

An r-rigid WJ-type model assumes that the radial coordinate r is frozen at some non-zero value r_0. Then, the radial degree of freedom can be suppressed, and the Hamiltonian in (53) reduces to [18]

$$H_{6DWJ} = \frac{\hbar^2}{2Mr_0^2}\Lambda^2 \tag{54}$$

Its eigenvalues determine the energies that are now not equidistant and are given by

$$\mathcal{E}(\upsilon) = \frac{\hbar^2}{2Mr_0^2}\upsilon(\upsilon+4) \equiv B\upsilon(\upsilon+4). \tag{55}$$

This expression produces a characteristic ratio $E_{4^+_1}/E_{2^+_1} \simeq 2.67$ (to be compared with the classical WJ value 2.5) of the ground state band energies with the subsidiary assumption $L = \upsilon$ (for example, the left diagonal of Table 1 of Ref. [18] with $(\lambda,\mu) = (k,0)$, $k = 0,2,4,\ldots$).

C. Rigid rotor submodel

Expression (45) can be rewritten in the form

$$\begin{aligned}\Lambda^2 =& 4\Big\{\frac{\partial^2}{\partial\alpha^2} + 2\cot 2\alpha\frac{\partial}{\partial\alpha} + \frac{1}{\sin^2\alpha}\frac{\partial^2}{\partial\phi^2} - \frac{\cos\alpha}{\sin^2\alpha}\overline{L}_z \mathrm{i}\frac{\partial}{\partial\phi}\Big\}\\ &-\Big[\Big(\frac{A(\alpha)+B(\alpha)}{2}\Big)\Big(\overline{L}^2-\overline{L}_z^2\Big)\\ &+\Big(\frac{A(\alpha)-B(\alpha)}{2}\Big)\Big(\overline{L}_x^2-\overline{L}_y^2\Big)+C(\alpha)\overline{L}_z^2\Big].\end{aligned} \tag{56}$$

When the other two collective variables α and ϕ are further frozen, the only remaining degrees of freedom are rotations in the three-dimensional space. The $SO(6)$ Casimir operator (56), with α and ϕ taking fixed values of α_0 and ϕ_0, reduces to

$$\begin{aligned}\Lambda^2 =& -\Big[\Big(\frac{A(\alpha_0)+B(\alpha_0)}{2}\Big)\Big(\overline{L}^2-\overline{L}_z^2\Big)\\ &+\Big(\frac{A(\alpha_0)-B(\alpha_0)}{2}\Big)\Big(\overline{L}_x^2-\overline{L}_y^2\Big)+C(\alpha_0)\overline{L}_z^2\Big]\Big\}.\end{aligned} \tag{57}$$

We can easily check that for small $\alpha_0 \to 0$, the corresponding factors become $(A(\alpha_0)+B(\alpha_0))/2 = 1/(1-\sin^2\alpha_0) \to 1$ and $(A(\alpha_0)-B(\alpha_0))/2 = -\sin\alpha_0/(1-\sin^2\alpha_0) \to 0$. The corresponding rotor-model Hamiltonian thus takes the form

$$H_{\rm rot}=\frac{\hbar^2}{2(\frac{1}{4}Mr_0^2)}\left(\overline{L}^2-\overline{L}_z^2\right)=\frac{\hbar^2}{2J}\left(\overline{L}^2-K^2\right)=a\left(\overline{L}^2-K^2\right),\quad(58)$$

where the moment of inertia is given by $J=Mr_0^2/4$, and the term $C(\alpha)\overline{L}_z^2$, pushed infinitely up in energy, is dropped. This corresponds to the case of axially symmetric nuclei, for which the moments of inertia in rotation about the symmetry axis are equal to zero, and hence the levels with projection $K\neq 0$ lie infinitely high in energy. The $SO(3)$ Casimir operator $\overline{L}^2$ in the intrinsic frame is defined in terms of the three Euler angles $g_3^+=(\theta_1,\theta_2,\theta_3)$ as

$$\overline{L}^2=\frac{1}{\sin\theta_1\cos\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin\theta_1\cos\theta_1\frac{\partial}{\partial\theta_1}\right)+\frac{1}{\cos^2\theta_1}\frac{\partial^2}{\partial\theta_2^2}+\frac{1}{\sin^2\theta_1}\frac{\partial^2}{\partial\theta_3^2}.\quad(59)$$

The eigenvalues of the rotor model Hamiltonian (58) are then simply

$$\mathcal{E}_K(L)=a\left[L(L+1)-K^2\right].\quad(60)$$

In this way, by considering the coordinate representation of the collective Hamiltonian, we have obtained the three exactly solvable submodel limits of the microscopic version of the BM model, which closely resemble the exact relationships of the original BM submodels (see, for example, Ref. [52]). We wish to point out that the three exactly solvable limits obtained are too simplistic, and in practical application to real nuclear systems, a more useful version is one in which the mixing of different $SU(3)$ irreducible representations within a single or different $SO(6)$ seniority representations is involved. Additionally, the (non-scalar) symplectic $Sp(12,R)$ bandhead structure $\langle\sigma\rangle\neq(0)$ for open-shell nuclei strongly affects the rigid-flow quadrupole dynamics. For instance, shell-model considerations based on the pseudo-$SU(3)$ [53] scheme give the $Sp(12,R)$ irreducible representation 0p-0h $[22]_6$ (or, equivalently, $\langle\sigma\rangle=\langle 65+191/2,43+191/2,\ldots,43+191/2\rangle$) for ^{192}Os [23]. The latter allows us to describe the low-energy quadrupole dynamics of an r-rigid WJ type in ^{192}Os via the mixing of different $SU(3)$ multiplets within the maximal seniority $SO(6)$ irrep $\upsilon_0=22$, as shown in Ref. [23]. This quadrupole dynamics is identically vanishing (or "frozen") within the scalar $\langle\sigma\rangle=(0)$ many-particle irreducible collective space of $Sp(12,R)$, corresponding to the phenomenological version of the original BM model. The given shell-model considerations demonstrate the crucial role played by the Pauli allowed subspaces $\mathbb{H}^{(\Lambda_0\omega\neq(0)\Gamma)}$ (with $\omega\equiv\sigma$) of the full many-particle Hilbert space in the description of the observed low-energy quadrupole dynamics in atomic nuclei.

VI. MATRIX REPRESENTATION

We want to stress that the extraction of collective and intrinsic wave functions, expressed in the coordinate representation, makes sense if we intend to solve the relevant dynamical equations for the collective functions. However, if we are not going to do this and want to use only the simplest kinematically correct wave functions provided by Eqs. (3) and (5) (or those equivalent to them in Eq. (7)), it is not necessary to represent them in the form of the sum of the products of the collective and intrinsic components. It is simpler to return from the collective and intrinsic variables to the initial Jacobi coordinates and use the wave functions $\Psi\binom{E_0E\upsilon\nu qLM}{\omega\delta[f]h}|x_1^a,\ldots,x_m^a)$, whose quantum numbers are defined by the reduction chains (3) and (5) (or those equivalent to them in Eq. (7)). Moreover, we can entirely avoid the coordinate representation and instead use the algebraic or matrix representation of the wave functions. In the matrix representation, the wave functions of the microscopic version of the BM model can simply be written as

$$|\Psi\left(E_0E\upsilon\nu qLM;\omega\delta[f]h\right)\rangle\equiv|E_0E\upsilon\nu qLM;\omega\delta[f]h\rangle,\quad(61)$$

where we recall that the symplectic bandhead σ (or the $Sp(12,R)$ irrep $\langle\sigma\rangle$) is determined by the $O(m)$ irreducible labels ω because $\sigma\equiv\omega$. Then, we can further construct the matrix representation of an arbitrary Hamiltonian in the basis (61).

Finally, note that by working in the matrix representation, we do not need the explicit dependence on the collective coordinates, and in this case, the construction of the basis states and the calculation of the required matrix elements, such as those of the radial function or spherical harmonics, can be performed in a purely algebraic way. For instance, the computational technique for performing realistic shell-model calculations within the microscopic version of the BM model has been given in Ref. [30].

VII. CONCLUSIONS

The recently proposed microscopic shell-model version of the BM collective model is considered in more detail in the coordinate representation. The latter possesses a clear and transparent physical meaning, which allows it to reveal several features of the new version of the collective model missed in the previous formulation. The relationship with the original BM model is considered, as well as between the different limiting submodels of the microscopic version of the BM model, which closely resemble the relationships of the original WJ and rotor models. The kinematically correct many-particle wave functions of the microscopic version of the BM model, conserving the experimentally observed integrals of motion, are shown to consist of collective irrotational-flow and intrinsic components – in accordance with the original BMU model. The general BM Hamiltonian is obtained as a contraction limit of the microscopic many-particle nuclear Hamiltonian, or, alternatively, by restricting the latter to the scalar $O(m)$ irreducible collective space $\mathbb{H}^{(\omega=(0))}$. We have demonstrated that this leads to some specific features (ignoring the many-fermion aspects) of the collective dynamics in the original phenomenological BM model, one of which is the violation of the Pauli principle, which leads to kinematically incorrect quantum-mechanical treatment of the nuclear many-particle system. The second is the ignoring of the intrinsic collective structure, determined by the non-zero $Sp(12,R)$ symplectic bandhead. In this way, by considering the Pauli allowed $Sp(12,R)$ irreducible representations, we recover the more complete quadrupole-monopole collective dynamics of many-particle proton-neutron nuclear systems. This more complete collective dynamics of atomic nuclei can thus be described within the framework of the well-defined many-nucleon quantum mechanics.

This study has a conceptual style and may appear too mathematical to many readers. Nevertheless, we point out that the present theory has already been successfully applied to several heavy mass nuclei with different collective properties, representing the typical collective structure characteristic for each of the various submodel limits of the BM model. However, we note that these first applications of the present symplectic based approach open the door for more intensive and systematic applications of this microscopic shell-model version of the BM model to different mass regions, which will allow for real systematics evaluations of the evolution of nuclear structure with the increase of the number of protons and neutrons. Especially interesting are the regions of transitional and weakly deformed nuclei, in which the different collective degrees of freedom are expected to be strongly coupled between themselves from one side and to the other non-collective (quasiparticle) degrees of freedom from another side. A complicated feature in these nuclei is the triaxial nature often observed in certain nuclear isotopes, which is usually described as γ-unstable in the WJ limit [19] of the BM model or as γ-rigid in the triaxial rotor model limit [20]. Of particular interest is the microscopic shell-model description of Cd isotopes, traditionally referred to as vibrational-like nuclei but recently shown to exhibit weakly deformed rigid- or irrotational-flow type quadrupole dynamics with the competition of intruder or shape coexistence structures.

Finally, we note that the results in this paper represent the so called algebraic approach to nuclear collective motion and, in particular, to the microscopic foundation of the BM model. We notice, however, that there is another line of theoretical development intensively exploited in the last two decades, which provides a microscopic foundation for the collective BM Hamiltonian of quadrupole vibrational and rotational degrees of freedom. The parameters of the five-dimensional collective Hamiltonian in these theoretical approaches, which are used to calculate the corresponding excitation energies and transition probabilities, are determined via constrained non-relativistic or relativistic self-consistent mean-field microscopic calculations. For instance, collective BM type Hamiltonians based on the relativistic energy density functionals have recently been developed and applied in a number of studies on nuclear structure phenomena related to, for example, the coupling of shape and pairing vibrations [54], quadrupole-octupole excitations [55], effects of triaxial deformation and dynamical correlations on the nuclear landscape [56], yrast and non-yrast excitations in neutron-rich ^{94,95,96}Kr isotopes [57], and intruder states and shape coexistence of Cd isotopes [58]. The exploited BM type collective Hamiltonians in these applications use more general and complicated expressions for the kinetic energy than the standard Bohr Hamiltonian of Eq. (31).

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