Charged Taub-NUT-de Sitter spacetime in DGP braneworld and its thermodynamics*

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Abstract: We study a charged Taub-NUT spacetime solution in the Dvali-Gabadadze-Porrati (DGP) brane. We show that the Reissner-Nordstrom-Taub-NUT-de Sitter solution of Einstein-Maxwell gravity solves the corresponding equations of motion, where the cosmological constant is related to the cross-over scale in the DGP model. Following the approach by Teitelboim in discussing the thermodynamics of de Sitter spacetime and the proposal by Wu *et al*. for a conserved charge associated with the NUT parameter, we obtain the generalized Smarr mass formula and the first law of thermodynamics of the spacetime.

Keywords: DGP brane, Taub-NUT, de Sitter, thermodynamics

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I. INTRODUCTION

The view that the universe is a brane embedded in higher dimensional space has attracted considerable attention in the last two decades. The absence of any experimental evidence for the existence of extra dimensions has not hindered theoretical investigations on braneworld theories. Indeed, superstring theory, which demands extra dimensions, seems to be compatible with the braneworld proposal [1]. Moreover, the discovery of gravitational waves and the black hole shadow has encouraged researchers to investigate these observations from the braneworld perspective [2–5].

There are several braneworld gravity models in literature, for example, the Horava-Witten [6], Arkani Hamed-Dimopoulos-Dvali (ADD) [7], Randall-Sundrum (RS) [8], and Dvali-Gabadadze-Porrati (DGP) [9] models. The DGP model, which is the topic of investigation in this study, has several interesting properties in regards to astronomical observations. It is well known that our universe is currently expanding, which can be explained by the positive value of the cosmological constant in Einstein's field equations. The positive cosmological constant is natural in the DGP brane model, where it can be connected to a cross-scale in the theory that governs the transition from five- to four-dimension. In other words, the observed self accelerating universe is well understood in the DGP brane model. This explains the steady interest in this subject, as seen from several recent works [10–16]. In particular, some charged black hole solutions in the DGP model have been reported [17–19].

In a model of gravity, in addition to the mass, charge, rotation, and even cosmological constant parameters, the so-called NUT parameter can also exist [20]. Despite obstacles related to spacetime for the NUT parameter, such as the closed timelike curve and conic singularity, recent investigations on this spacetime can still be found in literature. For example, the collisional Penrose process in Kerr-Taub-NUT spacetime is studied in [21], the thermodynamics related to NUT spacetime is explored in [22, 23], and discussions on M5 branes in Taub-NUT related space are presented in [24]. For the RS-II brane scenario, the related Taub-NUT spacetime has been discussed in [25]. It would be interesting to find a corresponding Taub-NUT related spacetime that fits in DGP brane theory and the corresponding thermodynamics. The DGP brane spacetime solutions presented in [17–19] have not yet incorporated the NUT parameter.

In this study, we pursue this aim, namely, finding an exact charged spacetime solution equipped with the NUT parameter, which solves the corresponding Hamiltonian constraint in the DGP model. The approach to solving the Hamiltonian constraint is similar to that used in Refs. [25–27] to find a black hole solution in the RS-II brane scenario. First, we employ the proper Kerr-Schild ansatz to solve the corresponding constraint and then apply an appropriate coordinate transformation to obtain the Boyer-Lindquist form of the metric. It turns out that there are two cases for the solutions of a charged Taub-NUT spacetime in the DGP brane. The first case corresponds to

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the vanishing Ricci scalar, and the second is associated with a constant Ricci scalar. The latter case is the Reissner-Nordstrom-Taub-NUT-de Sitter (RNTNdS) solution of Einstein-Maxwell theory. It is well known that studying the thermodynamic aspects of de Sitter spacetime is somewhat delicate owing to the existence of black hole and cosmological horizons in the spacetime. An interesting way to study the thermodynamics of de Sitter black hole spacetime was proposed by Teitelboim [28], in which we can have two different first laws of thermodynamics for each horizon. We can compute the Hawking temperature, entropy, angular velocity, and some other conjugate quantities for each horizon [29]. Because thermodynamics is discussed in regard to the black hole horizon, we consider the cosmological horizon as a boundary where the incorporated physical parameters are fixed. This is similar to the situation when one defines the mass in the ADM formalism where the boundary is at infinity. However, if we are investigating thermodynamics related to the cosmological horizon, the black hole horizon is considered a boundary with fixed parameters.

Moreover, the spacetime under investigation in this study is also equipped with the NUT parameter. The thermodynamics of Taub-NUT spacetime is also another problem to be discussed. Some previous studies have been dedicated to this, for example, the authors of [30] define the gravitational Misner charges to have the generalized first law in Taub-NUT spacetime. The authors of [31] studied the thermodynamics of Taub-NUT-AdS spacetime and showed that entropy can be obtained using the Noether charge method. An interesting proposal is given in Ref. [32], where the authors define a new conserved quantity associated with the NUT parameter. The proposal of the new conserved quantity $J_l = Ml$ in Ref. [32] resembles the angular momentum $J = Ma$ for Kerr spacetime. Using the new conserved quantity, the authors managed to obtain the generalized Smarr mass formula for the spacetime as well as the corresponding first law of thermodynamics.

In this paper, after showing that the RNTNdS solution solves the Hamiltonian constraints in the DGP brane, we study its thermodynamics. We adopt the approaches used by Teitelboim [28] and Sekiwa [29] to deal with the thermodynamic aspects of de Sitter spacetime, and the proposal by Wu *et al*. [32] to discuss the NUT parameter contribution. This paper is organized as follows. In the Sec. II, we review the DGP brane action and corresponding constraint equations. In Sec. III, we show that the Taub-NUT-de Sitter solution solves the equations of motion for the case with no electromagnetic fields on the brane. In Sec. IV, we extend the solution to a charged case, where we show that the RNTNdS solution can solve the constraints exactly. The thermodynamic aspects of the brane are discussed in Sec. V, and we present our conclusions in Sec. VI. In this paper, we use the natural units $G = c = \hbar = 1$.

II. ACTION AND EQUATION OF MOTION

In this chapter, we review the action and equations of motion associated with the DGP brane model. In the presence of sources, DGP gravitational action takes the form [9]

$$S = \tilde{M}^3 \int d^5x \sqrt{-\tilde{g}}\tilde{R} + \int d^4x \sqrt{-g}\left(M_P^2 R + L_{\rm matter}\right), \qquad (1)$$

where $\tilde{R}$ and R are the five- and four-dimensional Ricci scalars, respectively. In the above action, $L_{\rm matter}$ is a Lagrangian for matter localized on the brane. The five dimensional spacetime coordinates are denoted by $\tilde{x}^M = (x^\mu, x^4 = z)$, where $M = 0, 1, 2, 3, 4$, and $\mu = 0, 1, 2, 3$. The determinants of the five- and four-dimensional spacetime metric are given by $\tilde{g}$ and g, respectively, where the two metrics are related by $g_{\mu\nu} = \tilde{g}_{\mu\nu}(x^\mu, z = 0)$. We define the cross-over scale as $\gamma = 1/\lambda = M_P^2/2\tilde{M}^3$. Moreover, the boundary is at $z = 0$, and we assume $\mathbb{Z}_2$ symmetry across the boundary. Varying the action (1) with respect to the tensor metric $\tilde{g}_{MN}$ yields the equation of motion

$$\tilde{G}_{MN} = \tilde{\kappa}^2 \sqrt{\frac{g}{\tilde{g}}}\delta^\mu_M \delta^\nu_N \left(T_{\mu\nu} - \kappa^{-2}G_{\mu\nu}\right)\delta(z)\,, \qquad (2)$$

where $\tilde{G}_{MN} = \tilde{R}_{MN} - \frac{1}{2}\tilde{g}_{MN}\tilde{R}$ is the five-dimensional Einstein tensor, $G_{\mu\nu}$ is the four-dimensional Einstein tensor, R_{MN} is the five-dimensional Ricci tensor, $\kappa^2 = 1/M_P^2$, and $\tilde{\kappa}^2 = 1/\tilde{M}^3$. In the above equation, T_{MN} is the energy-momentum tensor in the bulk, whereas $T_{\mu\nu}$ is the energy-momentum tensor for matter localized on the brane.

We decompose the bulk metric into the following form:

$$\begin{aligned} ds^2 =& \tilde{g}_{MN}d\tilde{x}^M d\tilde{x}^N = \tilde{g}_{\mu\nu}(x,z)dx^\mu dx^\nu \\ &+ 2\mathcal{N}_\mu dx^\mu dz + \left(\mathcal{N}^2 + \tilde{g}_{\mu\nu}\mathcal{N}^\mu\mathcal{N}^\nu\right)dz^2\,. \end{aligned} \qquad (3)$$

Accordingly, the $\tilde{G}_{\mu z}$ and $\tilde{G}_{zz}$ components of (2) give the following equations [33]:

$$\tilde{\nabla}_\alpha K^\alpha_\mu - \tilde{\nabla}_\mu K = 0\,, \qquad (4)$$

and

$$R - K^2 + K_{\mu\nu}K^{\mu\nu} = 0\,. \qquad (5)$$

The extrinsic curvature tensor $K_{\mu\nu}$ is given by

$$K_{\mu\nu}=\frac{1}{2\mathcal{N}}\left(\partial_z\tilde{g}_{\mu\nu}-\tilde{\nabla}_\mu\mathcal{N}_\nu-\tilde{\nabla}_\nu\mathcal{N}_\mu\right), \tag{6}$$

and $\tilde{\nabla}_\mu$ is the covariant derivative associated with the metric tensor $\tilde{g}_{\mu\nu}$. Eqs. (4) and (5) are known as the momentum and Hamiltonian constraint equations, respectively. Moreover, the corresponding Israel's junction condition with $\mathbb{Z}_2$ symmetry can be obtained by integrating both sides of Eq. (2) along the z direction, followed by taking the limit $z\to 0$. This reads as

$$G_{\mu\nu}=\kappa^2T_{\mu\nu}+\lambda\left(K_{\mu\nu}-g_{\mu\nu}K\right). \tag{7}$$

If we consider the traceless $T_{\mu\nu}$ condition associated with the electromagnetic field trapped on the brane, Eq. (7) finds the momentum constraint (4) to be satisfied [18], whereas the Hamiltonian condition gives

$$R_{\mu\nu}R^{\mu\nu}+\lambda^2R-\frac{R^2}{3}+\kappa^4T_{\mu\nu}T^{\mu\nu}-2\kappa^2R_{\mu\nu}T^{\mu\nu}=0. \tag{8}$$

The equation of motion on the brane can be obtained by inserting Israel's junction condition (7) into the Einstein equations in the bulk, that is, $z\neq 0$, which reads as

$$\begin{aligned}R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+E_{\mu\nu}=&-\frac{\kappa^4}{\lambda^2}\left(T^\alpha_\mu T_{\alpha\nu}-\frac{1}{2}g_{\mu\nu}T_{\alpha\beta}T^{\alpha\beta}\right)\\&-\frac{1}{\lambda^2}\left(R^\alpha_\mu R_{\alpha\nu}-\frac{2}{3}RR_{\mu\nu}+\frac{1}{4}g_{\mu\nu}R^2-\frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}\right)\\&+\frac{\kappa^2}{\lambda^2}\left(R^\alpha_\mu T_{\alpha\nu}+T^\alpha_\mu R_{\alpha\nu}-\frac{2}{3}RT_{\mu\nu}-g_{\mu\nu}R_{\alpha\beta}T^{\alpha\beta}\right).\end{aligned} \tag{9}$$

In the above equation, $E_{\mu\nu}$ is the traceless "electric part" of the five-dimensional Weyl tensor $C_{\rm KLMN}$, and we set $\lambda=2\kappa^2/\tilde{\kappa}^2$ and consider $\kappa^2=8\pi$.

III. TAUB-NUT DGP BRANE

Let us first consider the vacuum case on the brane, namely, $T_{\mu\nu}=0$. For this, the Hamiltonian constraint (8) reduces to

$$R_{\mu\nu}R^{\mu\nu}-\frac{1}{3}R^2+\lambda^2R=0. \tag{10}$$

It turns out that this equation can be satisfied in two cases. The first case, which we refer to as the flat case, corresponds to the vanishing of the four-dimensional Ricci scalar and squared Ricci tensor,

$$R=0\quad,\quad R_{\mu\nu}R^{\mu\nu}=0. \tag{11}$$

In the second case, that is, the non-flat case, the related Ricci scalar and squared Ricci tensor satisfy

$$R=12\lambda^2\quad,\quad R_{\mu\nu}R^{\mu\nu}=36\lambda^4, \tag{12}$$

respectively.

In solving the Hamilton constraint condition (11), let us use the Kerr-Schild form for the metric

$$\mathrm{d}s^2=\mathrm{d}s^2_{\rm flat}+H(r,x)\left(k_\mu\mathrm{d}x^\mu\right)^2 \tag{13}$$

where

$$\begin{aligned}\mathrm{d}s^2_{\rm flat}=&-\left(\frac{r^2-l^2}{r^2+l^2}\right)\left(\mathrm{d}u^2+4lx\mathrm{d}\psi\mathrm{d}u\right)\\&+\frac{\Delta_x\left(r^2+l^2\right)^2-4l^2x^2\left(r^2-l^2\right)}{r^2+l^2}\mathrm{d}\psi^2\\&+\frac{r^2+l^2}{\Delta_x}\mathrm{d}x^2+2\mathrm{d}u\mathrm{d}r+4lx\mathrm{d}\psi\mathrm{d}r,\end{aligned} \tag{14}$$

$\Delta_x=1-x^2$, and $k_\mu\mathrm{d}x^\mu=\mathrm{d}u+2lx\mathrm{d}\psi$. Here, l denotes the NUT parameter. Accordingly, the corresponding Ricci scalar and square of the Ricci tensor related to metric (13) can be expressed as

$$R=\frac{\partial^2H(r,x)}{\partial r^2}+\frac{4r}{(r^2+l^2)}\frac{\partial H(r,x)}{\partial r}+\frac{2H(r,x)}{(r^2+l^2)}, \tag{15}$$

and

$$\begin{aligned}R_{\mu\nu}R^{\mu\nu}=&\frac{1}{2}\left(\frac{\partial^2H(r,x)}{\partial r^2}\right)^2+\frac{2}{(r^2+l^2)^2}\Big[r\left(r^2+l^2\right)\frac{\partial H(r,x)}{\partial r}\\&+2l^2H(r,x)\Big]\frac{\partial^2H(r,x)}{\partial r^2}+\frac{4r^2}{(r^2+l^2)^2}\left(\frac{\partial H(r,x)}{\partial r}\right)^2\\&+\frac{4rH(r,x)}{(r^2+l^2)^2}\frac{\partial H(r,x)}{\partial r}+\frac{2\left(r^4+5l^4-2l^2r^2\right)}{(r^2+l^2)^4}H(r,x)^2,\end{aligned} \tag{16}$$

respectively. The solution for $H(r,x)$ that satisfies both equations in (11) is given by[1)]

$$H(r,x)=\frac{2Mr}{r^2+l^2}. \tag{17}$$

Now, let us obtain the Boyer-Lindquist form of met-

1) Note that the vanishing of squared Ricci tensor $R_{\mu\nu}R^{\mu\nu}=0$ demands the absence of "tidal charge" β, which exists for the black hole in RS braneworld [34].

ric (13), with Eq. (17) as the $H(r,x)$ function. We can employ the following transformation:

$$\mathrm{d}u = \mathrm{d}t + \frac{r^2+l^2}{r^2-2Mr-l^2}\mathrm{d}r \ , \ \mathrm{d}\psi = \mathrm{d}\phi , \tag{18}$$

to the line element (13), and the result can be written as

$$\mathrm{d}s^2 = -\frac{r^2-2Mr-l^2}{r^2+l^2}(\mathrm{d}t+2lx\mathrm{d}\phi)^2 + \left(r^2+l^2\right)\left(\frac{\mathrm{d}r^2}{r^2-2Mr-l^2}+\frac{\mathrm{d}x^2}{\Delta_x}\right)+\left(r^2+l^2\right)\Delta_x \mathrm{d}\phi^2 . \tag{19}$$

The last equation is the well known Taub-NUT metric [20], which solves the vacuum Einstein equation. However, it is clear that the Taub-NUT solution solves constraint (11) because vacuum Einstein requires all the components of the Ricci tensor to be zero and obviously leads to the vanishing Ricci scalar.

For the non-flat case with a set of constraint equations given in (12), we can employ the Kerr-Schild metric as in (13) with the $\mathrm{d}s^2_{\rm flat} \to \mathrm{d}s^2_{\rm non-flat}$ replacement, that is,

$$\mathrm{d}s^2 = \mathrm{d}s^2_{\rm non-flat} + H(r,x)\left(k_\mu \mathrm{d}x^\mu\right)^2 , \tag{20}$$

where

$$\mathrm{d}s^2_{\rm non-flat} = 2\mathrm{d}u\mathrm{d}r + 4lx\mathrm{d}r\mathrm{d}\psi + \frac{\left(r^2+l^2\right)}{\Delta_x}\mathrm{d}x^2 - \frac{Z}{(r^2+l^2)}\mathrm{d}u^2 + \frac{\Delta_x\left(r^2+l^2\right)^2-4l^2x^2Z}{(r^2+l^2)}\mathrm{d}\psi^2 - \frac{4lxZ}{(r^2+l^2)}\mathrm{d}u\mathrm{d}\psi , \tag{21}$$

and

$$Z = r^2 - l^2 - \lambda^2\left(r^4+6r^2l^2-3l^4\right). \tag{22}$$

It turns out that this metric gives the Ricci scalar and square of the Ricci tensor exactly as indicated in (12) for $H(r,x)$, as in (17). To bring the line element (20) into the Boyer-Lindquist expression, we can employ the coordinate transformation

$$\mathrm{d}u = \mathrm{d}t + \frac{r^2+l^2}{Z-2Mr}\mathrm{d}r \ , \ \mathrm{d}\psi = \mathrm{d}\phi , \tag{23}$$

and the result is

$$\mathrm{d}s^2 = -\frac{Z-2Mr}{(r^2+l^2)}(\mathrm{d}t+2lx\mathrm{d}\phi)^2 + \left(r^2+l^2\right) \times\left(\frac{\mathrm{d}r^2}{Z-2Mr}+\frac{\mathrm{d}x^2}{\Delta_x}\right)+\left(r^2+l^2\right)\Delta_x\mathrm{d}\phi^2 . \tag{24}$$

The last equation can be understood as the Taub-NUT-de Sitter spacetime [20] with the cosmological constant $\Lambda = 3\lambda^2$. Obviously, this is a straightforward generalization of the solution from the flat case obeying the conditions in (11) to the non-flat case satisfying Eq. (12). Alternatively, we can directly use the ansatz for the flat case (13) to solve (12), where the result for the $H(r,x)$ function is

$$H(r,x) = \frac{2Mr+\lambda^2\left(r^4+6l^2r^2-3l^4\right)}{(r^2+l^2)} . \tag{25}$$

Inserting Eq. (25) into Eq. (13) yields exactly the same metric as in Eq. (20), with the $H(r,x)$ function as given in (17).

IV. CHARGED SOLUTION

Now that we have discussed several neutral solutions, let us turn to the electrically charged cases. Here, we consider the existence of source-free Maxwell fields outside the Taub-NUT black hole localized on the DGP brane obeying the equations of motion,

$$\nabla_\mu F^{\mu\nu} = 0 , \tag{26}$$

and the Bianchi identity $\nabla_{[\alpha}F_{\mu\nu]} = 0$. Note that the covariant derivative ∇_μ is defined with respect to the tensor metric $g_{\mu\nu}$ on the brane. In the presence of Maxwell fields, that is, $T_{\mu\nu} \neq 0$, the Hamiltonian constraint (8) can be solved by considering

$$R = 0 \ , \ R_{\alpha\beta}R^{\alpha\beta} = \frac{4Q^4}{(r^2+l^2)^4} \ ,$$

$$R_{\mu\nu}T^{\mu\nu} = \frac{Q^4}{2\pi(r^2+l^2)^4} \ , \ T_{\mu\nu}T^{\mu\nu} = \frac{Q^4}{16\pi^2(r^2+l^2)^4} , \tag{27}$$

for the flat case, and

$$R = 12\lambda^2 \ , \ R_{\mu\nu}T^{\mu\nu} = \frac{Q^4}{2\pi(r^2+l^2)^4} ,$$

$$R_{\alpha\beta}R^{\alpha\beta} = 36\lambda^4 + \frac{4Q^4}{(r^2+l^2)^4} \ , \ T_{\mu\nu}T^{\mu\nu} = \frac{Q^4}{16\pi^2(r^2+l^2)^4} , \tag{28}$$

for the non-flat case. Recall that the energy-momentum tensor related to the Maxwell fields is given by

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{29}$$

For the flat case, we find that the metric ansatz (13) with

$$H(r,x) = \frac{2Mr - Q^2}{(r^2+l^2)} \tag{30}$$

accompanied by the vector

$$A_\mu \mathrm{d}x^\mu = -\frac{Qr}{r^2+l^2}(\mathrm{d}u + 2lx\mathrm{d}\psi), \tag{31}$$

solves the constraints in (27). Again, the solution differs from that of the RS-II brane [25] family of black hole solutions, where it does not incorporate the "tidal charge," which is interpreted as an extra dimensional effect in the RS-II brane [26]. However, accompanied by the vector solution (31), the non-flat constraints (28) can be satisfied by the metric ansatz (20) with the corresponding $H(r,x)$ function given in Eq. (30). The corresponding Boyer-Lindquist form can be achieved using the coordinate transformation

$$\mathrm{d}u = \mathrm{d}t + \frac{r^2+l^2}{\Delta_Q}\mathrm{d}r\ ,\ \ \mathrm{d}\psi = \mathrm{d}\phi, \tag{32}$$

where $\Delta_Q = Z - 2Mr + Q^2$, and the Z function is given in (22). Accordingly, the resulting line element for a charged spacetime on the brane in the non-flat case can be written as

$$\begin{aligned}\mathrm{d}s^2 = &-\frac{\Delta_Q}{(r^2+l^2)}\mathrm{d}t^2 + \frac{\left(r^2+l^2\right)}{\Delta_Q}\mathrm{d}r^2 + \frac{\left(r^2+l^2\right)}{\Delta_x}\mathrm{d}x^2\\ &-\frac{4lx\Delta_Q}{(r^2+l^2)}\mathrm{d}t\mathrm{d}\phi + \frac{\left(r^2+l^2\right)^2\Delta_x - 4l^2x^2\Delta_Q}{(r^2+l^2)}\mathrm{d}\phi^2,\end{aligned} \tag{33}$$

which is simply the RNTNdS solution of Einstein-Maxwell theory with the cosmological constant $\Lambda = 3\lambda^2$. Metric (33) is accompanied by vector solution (31) in solving the corresponding non-flat charged Hamiltonian constraints (28).

So far, we have shown that the resulting spacetime metric on the brane is simply a family of RNTNdS type solutions in Einstein-Maxwell theory with the cosmological constant. However, in the corresponding equation of motion (9), there is a second rank tensor $E_{\mu\nu}$, which originates from the projection of the five-dimensional Weyl tensor C_{ABCD} on the brane [33, 35], that is, $E_{\mu\nu} = C_{ABCD}n^A n^C e^B_\mu e^D_\nu$. It turns out that this "electric part" of the five-dimensional Weyl tensor related to the charged solution above appears quite simple, where the non-zero parts can be obtained as

$$E_{tt} = -\frac{Q^2\Delta_Q}{(r^2+l^2)^3}, \tag{34}$$

$$E_{t\phi} = -\frac{2lxQ^2\Delta_Q}{(r^2+l^2)^3}, \tag{35}$$

$$E_{rr} = \frac{Q^2}{\Delta_Q(r^2+l^2)}, \tag{36}$$

$$E_{xx} = -\frac{Q^2}{\Delta_x(r^2+l^2)}, \tag{37}$$

$$E_{\phi\phi} = -\frac{Q^2\left(\left(r^2+l^2\right)^2\Delta_x + 4l^2x^2\Delta_Q\right)}{(r^2+l^2)^3}. \tag{38}$$

By assuming the existence of this second rank tensor $E_{\mu\nu}$, we can accept that the above metric and vector solution obey the Einstein equation, (9).

Obviously, the solutions presented here and in the previous section solve the Einstein equations

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 2F_{\mu\alpha}F^{\alpha}_{\nu} - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \tag{39}$$

Provided that the corresponding $E_{\mu\nu}$ does exist, these solutions can also solve the Einstein equation (9) and the Hamiltonian constraint (8). In fact, every solution of Eq. (39) solves the Hamiltonian constraint (8) and the Einstein equations on the brane (9) by assuming the existence of the associated $E_{\mu\nu}$.

V. THERMODYNAMICS OF THE NON-FLAT CASE

In the previous section, we show that the de Sitter family spacetime with $\Lambda = 3\lambda^2$ is a generic solution in the DGP brane for the non-flat case. A positive cosmological constant in the non-flat case can be explained from the cross-over scale between the bulk and the brane in DGP theory. In other words, the expansion of our universe can be considered as extra-dimensional effects according to this perspective. In this section, let us study the thermodynamics associated with a charged object with the NUT parameter in the non-flat case of the DGP brane, that is, the thermodynamics of RNTNdS spacetime. The line element under consideration is the one that appears in Eq. (33). As is typical for a black hole in de Sitter geometry, there are four zeroes in Δ_Q in Eq. (33), which are denoted by

$$r_{--} < r_- < r_+ < r_{++}\,. \tag{40}$$

The largest root r_{++} is acknowledged as the cosmological horizon r_c, whereas the smaller one r_+ is identified as the black hole horizon r_h. The existence of these two horizons is the origin of the problem in formulating the thermodynamical aspects of the spacetime. Each horizon is associated with a different temperature, and the two systems are not in thermal equilibrium. Furthermore, unlike in asymptotically flat spacetime, there is no notion of an observer at spatial infinity inside the cosmological horizon. Hence, defining the conserved quantities as the corresponding thermodynamical parameters requires special methods.

Here, we adopt the approach presented in Refs. [28, 36, 29] to investigate the thermodynamics of black holes in the de Sitter background. In this approach, because thermodynamics corresponding to the black hole horizon is discussed, the cosmological horizon is considered as just a boundary. This resembles the approach in an asymptotically flat black hole spacetime, where the ADM mass of the black hole is measured by an observer at the boundary, that is, at infinity. However, outside a black hole in de Sitter spacetime, there is no notion of spatial infinity because this point is located beyond the cosmological horizon [28, 29]. Conversely, when we study the thermodynamical aspects related to the cosmological horizon, the black hole horizon is considered the boundary. Reversal of these horizons yields the sign changes of the conserved quantities associated with the first law of thermodynamics with respect to each horizon, as we see below.

Related to black hole horizon thermodynamics, the conserved mass and electric charge in spacetime with line element (33) are given by

$$M_h = M\ ,\ Q_h = Q\,. \tag{41}$$

Above, the subscript "h" refers to the quantities related to the black hole horizon. However, the mass and electric charge associated with the cosmological horizon are given by

$$M_c = -M\ ,\ Q_c = -Q\,, \tag{42}$$

where the subscript "c" denotes the parameters corresponding to the cosmological horizon. Indeed, in classical Einstein gravity, the cosmological constant is just a constant, and one cannot consider it to vary in a thermodynamical relation. However, as argued in Refs. [37, 38], the cosmological constant can be viewed as a thermodynamical variable in a semiclassical approach. Here, we treat the cosmological constant as a variable in thermodynamics and define the physical cosmological constant parameters as [29]

$$\Lambda_h = 3\lambda^2\ ,\ \Lambda_c = -3\lambda^2\,. \tag{43}$$

Intuitively, Λ_h corresponds to the black hole horizon, and Λ_c is associated with the cosmological one. In addition to the above parameters, we add more conserved quantities to the system, which are related to the NUT parameter. Following the prescriptions in Ref. [32], we define

$$N_h = Ml\ ,\ N_c = -Ml\,, \tag{44}$$

as the new quantities associated with the black hole and cosmological horizons, respectively. The definitions of these new quantities are analogous to the conserved angular momentum in Kerr spacetime, that is, $J = Ma$, where the NUT parameter l plays a similar role to the rotational parameter a in rotating spacetime. Clearly, we do not provide any surface integrals to obtain the conserved quantities in (44). Indeed, there are several studies that can be used to construct such integrals to obtain a charge associated with the NUT parameter, such as the covariant phase space approach [39, 40]. Particularly, for asymptotically local de Sitter spacetimes, discussions on conservative charges can be found in Refs. [41–44]. Despite the absence of a surface integral formula for the conservative charge related to the NUT parameter, the authors of [32] have shown that defining $N = Ml$ leads to a nice generalized Smarr mass formula and can produce the expected entropy as a quarter of the black hole area. Moreover, several previous studies support this proposal, For example, as noted by the authors of [45], $N = Ml \equiv M_5$ can be viewed as a conserved mass of a five-dimensional magnetic monopole. Another supporting result is the work presented in Ref. [27], where the definition $N = Ml$ can explain the gyromagnetic ratio of Kerr-Taub-NUT spacetime. Therefore, we can expect the definition in (44) to give us some good results, even though we now have both the black hole and cosmological horizons instead of only the black hole horizons as in Ref. [32].

In addition to the quantities defined in Eq. (44), which are treated as thermodynamic variables, the NUT parameter l itself should be regarded as another variable in thermodynamics to yield a consistent Bekenstein-Smarr mass formula. The Misner string attached at the south and north poles can carry some rotation-like and electromagnetic-like energies [32]. It is well known that the first law of thermodynamics for Kerr-Newman black holes can be written as

$$dM = TdS + \Omega_h dJ + \Phi_h dQ, \tag{45}$$

where M is the black hole mass, T is the Hawking temperature, S is the black hole entropy, Ω_h is the angular velocity at the horizon, J is the angular momentum, Q is the black hole charge, and Φ_h is the electric potential at the horizon. In the last equation, the change in the rotational energy of the Kerr-Newman black hole is represented by the term $\Omega_h dJ$, whereas the contribution of electromagnetic energy is given by the term $\Phi_h dQ$.

In this study, we consider that the first law of thermodynamics for the Taub-NUT spacetime takes a form similar to that in Eq. (45), namely,

$$dM = TdS + \omega_h dN + \varpi_h dl, \tag{46}$$

where the change in rotation-like energy is given by the term $\omega_h dN$, and the electromagnetic-like energy contribution can be found in the term $\varpi_h dl$. The definition of $N = Ml$, which resembles the angular momentum $J = Ma$, and the fact that the NUT parameter l is viewed as some conserved quantity similar to Q in the Kerr-Newman case make the correspondence between Eqs. (45) and (46) clear. In Taub-NUT spacetime, we have the NUT potential at the horizon ϖ_h, which acts as the electric potential at the horizon Φ_h in Kerr-Newman spacetime. Later, it can be shown that the first law of thermodynamics that takes the form in (46) can produce the mass formula

$$M = 2TS + 2\omega N + \varpi l. \tag{47}$$

However, several subtleties emerge here that are related to the NUT parameter as a conserved quantity. Indeed, so that the NUT parameter l can act exactly as the electric charge of a body in de Sitter spacetime, we should have $l_h = -l_c = l$. However, this setting leads to an inconsistency in the definition of the conserved quantities $N = Ml$ with respect to the black hole and cosmological horizons, as given in (44). Recall that the quantity $N = Ml$ is inspired by the angular momentum $J = Ma$, and for a rotating object in de Sitter spacetime, we have the relation $J_h = -J_c = Ma$ [29]. Here, we maintain this resemblance between N and J, that is, they both change sign as the horizons are reversed. Therefore, we must consider the NUT parameter as a conserved quantity that does not change sign as the horizons are reversed in their role, that is,

$$l_h = l = l_c. \tag{48}$$

This is similar to the invariance of the rotational parameter a under the change in the horizon's role in Kerr-de Sitter spacetime [29].

A. Black hole horizon thermodynamics

In this section, we study the thermodynamics of the black hole horizon, where the incorporated physical parameters are

$$M_h = M\ ,\ N_h = Ml\ ,\ Q_h = Q\ ,\ l_h = l\ ,\ \Lambda_h = 3\lambda^2. \tag{49}$$

The area of the RNTNdS horizon is $A_h = 4\pi\Xi_h$, where $\Xi_h = r_h^2 + l^2$. From the condition $\Delta_Q(r_h) = 0$, we can have the following relation:

$$\Xi_h = 2M_h r_h + 2l^2 - Q_h^2 + \lambda^2\left(r_h^4 + 6l^2 r_h^2 - 3l^4\right). \tag{50}$$

Furthermore, the last equation can be rearranged to

$$\left(1 - 4\lambda^2 l^2\right)\left(\Xi_h - 2l^2\right) + Q_h^2 - \lambda^2\Xi_h^2 = 2M_h r_h. \tag{51}$$

It turns out that adding N_h^2 to the square of Eq. (51) can give the squared mass formula

$$4M_h^2\Xi_h = 4N_h^2 + \left\{\left(1 - 4\lambda^2 l^2\right)\left(\Xi_h - 2l^2\right) + Q^2 - \lambda^2\Xi_h^2\right\}^2, \tag{52}$$

which can be rewritten in the following form:

$$M_h^2 = \frac{1}{4\Xi_h}\left\{\left(1 - \frac{32\pi}{3}\mathcal{P}_h l^2\right)\left(\Xi_h - 2l^2\right) + Q_h^2 - \frac{8\pi}{3}\mathcal{P}_h\Xi_h^2\right\}^2 + \frac{N_h^2}{\Xi_h}. \tag{53}$$

The last expression is the generalized Smarr formula associated with the black hole horizon. In the above equation, we consider the cosmological constant $\Lambda_h = 3\lambda^2$ to be a thermodynamic variable, which consequently causes the generalized pressure $\mathcal{P}_h = \Lambda_h(8\pi)^{-1}$ to also be a thermodynamic variable. Accordingly, the first law of thermodynamics can be obtained as

$$dM_h = \frac{\kappa_h}{8\pi}dA_h + \omega_h dN_h + \varpi_h dl_h + \Phi_h dQ_h + \mathcal{V}_h d\mathcal{P}_h, \tag{54}$$

where the surface gravity, NUT potential, conjugate thermodynamic volume, electrostatic potential, and angular velocity at the black hole horizon are given by

$$\kappa_h = \frac{r_h - M_h - 2\lambda^2 r_h\left(r_h + 3l^2\right)}{\Xi_h}, \tag{55}$$

$$\varpi_h = 2lr_h - \frac{1 + 2\lambda^2\left(r_h^2 - 3l^2\right)}{\Xi_h}, \tag{56}$$

$$\mathcal{V}_h = -\frac{4\pi r_h\left(r_h^4 + 6l^2 r_h^2 - 3l^4\right)}{3\Xi_h}, \tag{57}$$

$$\Phi_h = \frac{Q_h r_h}{r_h^2 + l^2}, \tag{58}$$

and

$$\omega_h = \frac{l}{\Xi_h}, \tag{59}$$

respectively. We can understand that Eq. (54) reflects the conservation of black hole energy, where it now incorporates the variations in the NUT parameter l_h and the conserved quantity N_h associated with it. Furthermore, using the above results, the Bekenstein-Smarr mass formula can be verified as

$$M_h = 2T_h S_h + 2\omega_h N_h + \varpi_h l_h + \Phi_h Q_h - 2\mathcal{V}_h \mathcal{P}_h, \tag{60}$$

where the Hawking temperature is given by $T_h = \kappa_h/2\pi$. In the last equation, the physical parameters M_h, N_h, l_h, Q_h, and P_h are the fixed conserved quantities measured at the cosmological horizon as the boundary, which are related to the quantities T_h, S_h, ω_h, ϖ_h, Φ_h, and $\mathcal{V}_h$ evaluated at the black hole horizon.

B. Cosmological horizon thermodynamics

Now, let us turn to the thermodynamic aspects associated with the cosmological horizon. The prescription is similar to the previous one on black hole horizon thermodynamics; however, we must switch the role of the horizons. In contrast with the previous subsection, where we consider the cosmological horizon to be just a boundary, the black hole horizon now plays the role of a boundary. The incorporated physical parameters are

$$M_c = -M\ ,\ N_c = -Ml\ ,\ Q_c = -Q\ ,$$

$$l_c = l\ ,\ \Lambda_c = -3\lambda^2. \tag{61}$$

From the last equation, we can have $N_c = -N_h$ as an implication that the NUT parameter does not change sign as the horizons are switched. It eventually has a significant impact when we attempt to express the relation between the first laws of thermodynamics corresponding to each horizon in spacetime.

The area of the cosmological horizon in RNTNdS spacetime (33) is given by $A_c = 4\pi\Xi_c$, where $\Xi_c = r_c^2 + l^2$. Likewise, the condition $\Delta_Q(r_c) = 0$ can give us

$$\Xi_c = 2Mr_c + 2l^2 - Q_c^2 + \lambda^2\left(r_c^4 + 6l^2 r_c^2 - 3l^4\right), \tag{62}$$

and similar to the black hole horizon case, we can have

$$4{M_c}^2\Xi_c = 4N_c^2 + \left\{\left(1 - 4\lambda^2 l^2\right)\left(\Xi_c - 2l^2\right) + Q^2 - \lambda^2\Xi_c^2\right\}^2. \tag{63}$$

An arrangement of the last equation can produce

$$\begin{aligned}{M_c}^2 = &\frac{1}{4\Xi_c}\left\{\left(1 - \frac{32\pi}{3}\mathcal{P}_c l^2\right)\left(\Xi_c - 2l^2\right)\right.\\ &\left. + Q_c^2 - \frac{8\pi}{3}\mathcal{P}_c\Xi_c^2\right\}^2 + \frac{N_c^2}{\Xi_c},\end{aligned} \tag{64}$$

where the thermodynamic variable $\mathcal{P}_c = \Lambda_c(8\pi)^{-1}$ is the generalized pressure corresponding to the cosmological horizon.

Consequently, the mass formula (64) corresponds to the first law of thermodynamics

$$\mathrm{d}M_c = \frac{\kappa_c}{8\pi}\mathrm{d}A_c + \omega_c\mathrm{d}N_c + \varpi_c\mathrm{d}l_c + \Phi_c\mathrm{d}Q_c + \mathcal{V}_c\mathrm{d}\mathcal{P}_c, \tag{65}$$

where the surface gravity, NUT potential, conjugate thermodynamic volume, electrostatic potential, and angular velocity at the cosmological horizon are given by

$$\kappa_c = \frac{r_c - M_c - 2\lambda^2 r_c\left(r_c + 3l^2\right)}{\Xi_c}, \tag{66}$$

$$\varpi_c = 2lr_c - \frac{1 + 2\lambda^2\left(r_c^2 - 3l^2\right)}{\Xi_c}, \tag{67}$$

$$\mathcal{V}_c = -\frac{4\pi r_c\left(r_c^4 + 6l^2 r_c^2 - 3l^4\right)}{3\Xi_c}, \tag{68}$$

$$\Phi_c = -\frac{Q_c r_c}{r_c^2 + l^2}, \tag{69}$$

and

$$\omega_c = \frac{l}{\Xi_c}, \tag{70}$$

respectively. If Eq. (54) can be interpreted as the law of conservation energy related to the black hole horizon, Eq. (65) reflects the conservation of energy inside the cosmological horizon. Because the spacetime possesses an NUT parameter l_c, the total energy inside the cosmological horizon obtains contributions from changes in the rotation-like and electromagnetic-like energies originating from this NUT parameter. Using the above results, the generalized Smarr mass formula

$$M_c = 2T_c S_c + 2\omega_c N_c + \varpi_c l + \Phi_c Q_c - 2\mathcal{V}_c \mathcal{P}_c \tag{71}$$

can be justified. This result reveals that the total energy inside the cosmological horizon can be accessed by an observer outside the black hole horizon.

As the author of [29] showed, the relations between the physical parameters associated with each horizon allow one to connect the corresponding first law of thermodynamics. Using the equations $\mathrm{d}M_h = -\mathrm{d}M_c$, $\mathrm{d}N_h = -\mathrm{d}N_c$, $\mathrm{d}l_h = \mathrm{d}l_c$, $\mathrm{d}Q_h = -\mathrm{d}Q_c$, and $\mathrm{d}\mathcal{P}_h = -\mathrm{d}\mathcal{P}_c$, adding Eq. (54) to Eq. (65) can give us

$$\begin{aligned}&(\mathcal{V}_h - \mathcal{V}_c)\,\mathrm{d}(\mathcal{P}_c) + (\Phi_h - \Phi_c)\,\mathrm{d}(Q_c) + (\omega_h - \omega_c)\,\mathrm{d}(N_c)\\ &-(\varpi_h + \varpi_c)\,\mathrm{d}(l_c) - T_h \mathrm{d}S_h = T_c \mathrm{d}S_c.\end{aligned} \tag{72}$$

Basically, Eq. (72) is similar to the result presented in Ref. [29] for the black hole horizon and cosmological horizon entropy relation in the Kerr-Newman-de Sitter spacetime

$$\begin{aligned}&(\mathcal{V}_c - \mathcal{V}_h)\,\mathrm{d}(-\mathcal{P}_c) + (\Phi_c - \Phi_h)\,\mathrm{d}(-Q_c)\\ &+(\Omega_c - \Omega_h)\,\mathrm{d}(-J_c) - T_h \mathrm{d}S_h = T_c \mathrm{d}S_c,\end{aligned} \tag{73}$$

except the term that contains the sum of the NUT potentials ϖ_c and ϖ_h. Note that the conserved angular momentum $J_h = -J_c$ in Eq. (73) corresponds to the rotational energies in the rotating spacetime. However, the quantity $N_h = -N_c$ in Eq. (72) exists due to the presence of the NUT parameter in the spacetime, which can be interpreted owing to the existence of the Misner string attached to the south and north poles. In another study, the authors showed how to associate this Misner string to entropy, which can then be incorporated into the corresponding first law of thermodynamics [30].

To extract the physical meaning of Eq. (72), let us compare it to a similar equation given in (73). It can be easily observed that the first three terms on the l.h.s. of both equations are analogous. The first term reflects the increase in vacuum energy inside the cosmological horizon, the second term describes the extracted electromagnetic energy[1)], and the third term represents the extracted rotational or rotational-like energy. However, the fourth term on the l.h.s of Eq. (72) has no analogous counterpart in the first law of Kerr-Newman-de Sitter black hole thermodynamics (73). We interpret this fourth term as the total electromagnetic-like energy inside the cosmological horizon, originating from the presence of the Misner string, which extends in the visible region $r_h < r < r_c$, that is, from the black hole surface to the cosmological horizon. Finally, the fifth term on the l.h.s. of Eq. (72) and also the fourth term on the l.h.s. of Eq. (73) are related to the Hawking radiation of the black hole. From Eq. (72), we can learn that the energy decrease in the visible region and the increase in the black hole mass would lead to a decrease in the cosmological horizon entropy.

Before we reach our conclusion, let us remark on the obtained results in this section. Indeed, the Bekenstein-Smarr mass formula (60) can also be applied to the horizon of the Reissner-Nordstrom-Taub-NUT-anti de Sitter (RNTNAdS) black hole, as discussed in [32]. This can be performed by replacing $\lambda^2 \to -\lambda^2$ in the corresponding equations. Particularly, the generalized pressure $\mathcal{P}_h$ changes sign in Eq. (54), which reflects a different way for the total mass to change with respect to the variation in the generalized pressure $\mathcal{P}_h$. Surely, the RNTNAdS spacetime does not solve the equations of motion in the DGP brane presented in Sec. II because it corresponds to the Einstein equations with a negative cosmological constant. Moreover, cosmological horizon does not exist in anti-de Sitter spacetime. Therefore, the results obtained above for cosmological horizon thermodynamics and the related relation in Eq. (72) that connects two entropies do not apply for RNTNAdS spacetime.

VI. CONCLUSION

In this paper, we show that the RNTNdS spacetime of Einstein-Maxwell theory exactly solves the non-flat case of the four dimensional equations of motion in the DGP braneworld scenario, provided a particular form of the five dimensional Weyl tensor projection is satisfied. This approach in solving the equations of motion in a braneworld theory has been applied to several cases, including a recent one in the RS-II brane [25]. However, unlike the spacetime solution in the RS-II braneworld, which can include the so called tidal charge, the Hamiltonian constraint conditions in DGP theory cannot allow such charge to exist.

The thermodynamics of RNTNdS spacetime on the

1) Note that the RNTNdS spacetime possesses the electric charge Q as well.

DGP brane is discussed in Sec. V. The positive cosmological constant is understood to be related to the cross-over scale in the theory. By adopting the approach by Sekiwa [29] in discussing the black hole thermodynamics in de Sitter spacetime, and the proposal by Wu *et al*. [32] in defining a new conserved quantity associated with the NUT parameter, we manage to express the generalized Smarr formula and first law of thermodynamics for each horizon in RNTNdS spacetime. Interestingly, the first law of thermodynamics for each horizon can be combined to give an equation that describes the change in cosmological horizon entropy with respect to the increase in the black hole mass and the energies in the visible region. This result is novel because no previous studies have addressed the thermodynamics of Taub-NUT de Sitter spacetime in this fashion.

However, we do not consider the Gibbs energy associated with the spacetime in this paper. As shown in Ref. [38], the Gibbs energy can be an effective tool to study the thermodynamics of de Sitter spacetime. Moreover, we can also add the inner horizon entropy that can contribute to the total first law of thermodynamics in the system. Incorporating rotation can also be an interesting addition to the solution, as calculated in Sec. IV, or the thermodynamics investigation, as presented in Sec. V. Moreover, insight from the AdS/CFT correspondence shows that considering the cosmological constant as a thermodynamic variable is not the correct way to obtain the laws of spacetime thermodynamics [46]. To do so, one should use the Noether-Wald method [40], which relies on asymptotic charges. These problems can be included in our future work.

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