Further study on lepton mass spectra and flavor mixing with $S_{3L} \times S_{3R}$ flavor symmetry[*]

Chang-Qi Hu (胡倡旗)[†] Chun-Yuan Li (李春园)[‡] Xiao Liang (梁霄)[§]
Xing-Hua Yang (杨兴华)[♮] Dai-Xing Zhang (张代兴)[♯]

School of Physics and Optoelectronic Engineering, Shandong University of Technology, Zibo 255000, China

Abstract: Neutrino oscillation experiments have confirmed that neutrinos are massive particles and that lepton flavors are mixed. To explain the observed lepton mass spectra and flavor mixing patterns, flavor symmetry plays a crucial and unique role. In this paper, we propose a useful symmetry-breaking scheme by applying $S_{3L} \times S_{3R} \to S_{2L} \times S_{2R} \to \emptyset$ within both charged-lepton and neutrino sectors at the mass-matrix level. For the three distinct residual subgroups $S_{2L}^{(23)} \times S_{2R}^{(23)}$, $S_{2L}^{(13)} \times S_{2R}^{(13)}$, and $S_{2L}^{(12)} \times S_{2R}^{(12)}$ that are under consideration, we systematically analyze the various parameterizations of the lepton mass matrices. It is shown that all three scenarios are in good agreement with current neutrino oscillation data. Notably, within the latest best-fit values of neutrino oscillation parameters, the predicted values of the Dirac CP-violating phase δ are $294.6°$, $302.3°$, and $287.0°$, respectively. To further assess the viability of the model, a comprehensive numerical analysis is performed by utilizing neutrino oscillation parameters at the 3σ level. It is found that the allowed ranges of δ are $281.2° \to 338.7°$, $287.0° \to 342.2°$, and $282.7° \to 297.0°$, which all fall within its 3σ range. These results indicate that the proposed symmetry-breaking scheme $S_{3L} \times S_{3R} \to S_{2L} \times S_{2R} \to \emptyset$ can naturally explain the realistic lepton mass hierarchy and mixing pattern, thereby providing valuable theoretical perspectives for future research.

Keywords: $S_{3L} \times S_{3R}$ flavor symmetry, lepton flavor mixing, CP violation

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I. INTRODUCTION

The standard model is one of the most successful theories in physics, providing a precise and unified description of fundamental particles and their interactions [1−3]. However, several problems remain unresolved [4−7]. Among these open questions, the phenomena of neutrino oscillations provide the most direct evidence of new physics beyond the standard model [8]. Within the framework of the standard model, the absence of right-handed neutrino fields prevents the generation of Dirac mass terms through Yukawa couplings, and thus neutrinos remain strictly massless. In recent years, a series of neutrino oscillation experiments have revealed that neutrinos are massive particles and that lepton flavors are mixed, which has stimulated extensive research on new physics beyond the standard model; see Ref. [9] for a recent review.

Recently, neutrino oscillation experiments, such as SNO+ [10], Super-Kamiokande [11], T2K [12], and NOνA [13] have significantly improved the precision of neutrino oscillation parameter measurements, including three mixing angles ($\theta_{12}, \theta_{23}, \theta_{13}$) and two mass-squared differences ($\Delta m_{21}^2, \Delta m_{31}^2$). The global analysis of neutrino oscillation data also provides constraints on the observable parameters [14−16]. For instance, the latest best-fit values of three neutrino mixing angles provided by NuFIT 6.0 (2024) [14] are

$$\theta_{12} = 33.68^{+0.73°}_{-0.70°}, \quad \theta_{23} = 48.5^{+0.7°}_{-0.9°}, \quad \theta_{13} = 8.52^{+0.11°}_{-0.11°}. \quad (1)$$

In case of the normal neutrino mass hierarchy ($m_1 <$

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[†] E-mail: 23412011092@stumail.sdut.edu.cn
[‡] E-mail: lichunyuan@sdut.edu.cn
[§] E-mail: lx321@sdut.edu.cn
[♮] E-mail: yangxinghua@sdut.edu.cn
[♯] E-mail: 24412011130@stumail.sdut.edu.cn

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$m_2 < m_3$), the best-fit values of two neutrino mass-squared differences are

$$\Delta m_{21}^2 = (7.49^{+0.19}_{-0.19}) \times 10^{-5}\ \text{eV}^2\,,$$
$$\Delta m_{31}^2 = (2.534^{+0.025}_{-0.023}) \times 10^{-3}\ \text{eV}^2\,. \qquad (2)$$

While the neutrino mass-squared differences are measured with ever-increasing precision, the absolute scale of neutrino masses has not yet been determined. The KATRIN experiment recently performed precision spectroscopy of the tritium β-decay close to the kinematic endpoint and set an upper limit on the effective electron antineutrino mass of $m_\nu < 0.45$ eV at 90% confidence level [17].

The above discussions strongly indicate that the generation mechanism of neutrino masses is quite different from that of other fermions in the standard model. At present, various mechanisms for neutrino mass generation have been proposed, such as the canonical seesaw mechanism [18–22], inverse seesaw mechanism [23, 24], and radiative mass generation [25–28]. While these mechanisms are effective in generating small neutrino masses, they generally lack the ability to constrain the flavor structures of massive neutrinos, which usually requires additional flavor symmetries [29, 30]. The $S_{3L}\times S_{3R}$ symmetry has been widely studied, as it predicts that the Yukawa interactions of charged-fermions take the well-known "democratic" matrix form [31–68]. Furthermore, to account for the mass spectra and flavor mixing, a variety of different symmetry-breaking patterns have been proposed.

In this study, we adopt a phenomenological approach and work at the mass-matrix level. To explain the realistic lepton mass spectra and mixing angles, we apply the permutation symmetry S_3 to both charged-lepton and neutrino sectors. Different from previous studies, we propose an interesting scheme to break the $S_{3L}\times S_{3R}$ flavor symmetry, where the same two-stage symmetry-breaking chain $S_{3L}\times S_{3R} \to S_{2L}\times S_{2R} \to \emptyset$ is applied in parallel to the charged-lepton and neutrino mass matrices. There are a total of nine parameters in the model, which can be fully determined by nine corresponding observables. Moreover, the model enables the predictions of the Dirac and Majorana CP-violating phases, the sum of neutrino masses, and the effective neutrino mass, as well as other relevant quantities.

The rest of this paper is organized as follows. A theoretical framework based on $S_{3L}\times S_{3R}$ flavor symmetry is briefly introduced in Sec. II. In Sec. III, we specify the three distinct symmetry-breaking chains and explore their implications for the lepton mass spectra and flavor mixing. Sec. IV details the numerical determination of model parameters. Finally, we present a brief summary in Sec. V.

II. $S_{3L}\times S_{3R}$ FLAVOR SYMMETRY FRAMEWORK

As indicated in Ref. [53], from a phenomenological perspective, the Lagrangian relevant to lepton masses at low energies can be expressed as

$$-\mathcal{L}_{\rm mass} = \overline{l_L} M_l l_R + \frac{1}{2}\overline{\nu_L} M_\nu (\nu_L^c) + \text{h.c.}\,, \qquad (3)$$

where M_l denotes the mass matrix of the charged-leptons, and M_ν represents the effective Majorana neutrino mass matrix. The latter may naturally arise from various neutrino mass models. To account for the experimentally observed lepton mass spectra and flavor mixing patterns, we reconsider a simple model based on the discrete symmetry $S_{3L}\times S_{3R}$. Here, the subscripts L and R denote the transformation properties under the left- and right-handed flavor symmetries, respectively. S_3 denotes the permutation group of three objects, and its three-dimensional reducible representation can be expressed as

$$S^{(123)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S^{(231)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
$$S^{(312)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^{(213)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$S^{(132)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^{(321)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \qquad (4)$$

Assuming that the Lagrangian in Eq. (3) remains invariant under $S_{3L}\times S_{3R}$ symmetry, the mass matrices of the charged-leptons and neutrinos should satisfy the following relations:

$$S_{3L} M_l = M_l S_{3R}\,, \quad S_{3L} M_\nu = M_\nu S_{3L}\,. \qquad (5)$$

Specifically, in the limit of $S_{3L}\times S_{3R}$ symmetry, the charged-lepton mass matrix takes the so-called democratic form

$$M_l^{(0)} = \frac{c_l}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad (6)$$

where c_l measures the mass scale of charged-leptons. Similarly, the neutrino mass matrix in the symmetry limit can be expressed as

$$M_\nu^{(0)}=c_\nu\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+r_\nu\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\right],\qquad(7)$$

where c_ν sets the mass scale of neutrinos, and r_ν quantifies the deviation from the identity matrix. It is worth mentioning that, to obtain two large lepton mixing angles, $|r_\nu|\ll 1$ must be satisfied. Note that both $M_l^{(0)}$ and $M_\nu^{(0)}$ can be diagonalized by the same orthogonal matrix as follows:

$$V_D^T M_l^{(0)} V_D=\frac{c_l}{3}\begin{pmatrix}0&0&0\\0&0&0\\0&0&3\end{pmatrix},$$

$$V_D^T M_\nu^{(0)} V_D=c_\nu\begin{pmatrix}1&0&0\\0&1&0\\0&0&1+3r_\nu\end{pmatrix},$$

$$\text{with }\ V_D=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\0&-\frac{2}{\sqrt6}&\frac{1}{\sqrt3}\end{pmatrix}.\qquad(8)$$

It is evident from Eq. (8) that only the third-generation charged-lepton acquires a non-zero mass, and the first two generations remain massless. In the neutrino sector, the three generation neutrinos are nearly degenerate. Furthermore, the lepton mixing matrix in this case turns out to be an identity matrix. Hence, it is imperative to break the $S_{3L}\times S_{3R}$ symmetry in the charged-lepton and neutrino mass matrices to explain the realistic lepton masses and mixing angles.

In this study, we assume that the $S_{3L}\times S_{3R}$ symmetry may first be broken to a residual $S_{2L}\times S_{2R}$ symmetry, followed by random perturbations to entirely break the remaining symmetry. The two-stage symmetry-breaking scheme is applied to both charged-lepton and neutrino mass matrices in a similar way. According to the above discussions, the mass matrices of the charged-leptons and neutrinos can then be decomposed as

$$M_l=M_l^{(0)}+\Delta M_l^{(1)}+\Delta M_l^{(2)},\qquad(9)$$

$$M_\nu=M_\nu^{(0)}+\Delta M_\nu^{(1)}+\Delta M_\nu^{(2)}.\qquad(10)$$

Here, $M_l^{(0)}$ and $M_\nu^{(0)}$ are the symmetry-limit terms determined by the $S_{3L}\times S_{3R}$ symmetry, as shown in Eq. (6) and Eq. (7). $\Delta M_l^{(1)}$ and $\Delta M_\nu^{(1)}$ are the first-order perturbation terms controlled by the residual $S_{2L}\times S_{2R}$ symmetry, while $\Delta M_l^{(2)}$ and $\Delta M_\nu^{(2)}$ are the second-order perturbation terms that are employed to eventually break the remaining symmetry. The specific form of the perturbation terms will be shown later.

III. LEPTON MASS SPECTRA AND MIXING MATRIX

Based on the symmetry-breaking hypothesis proposed above, the first-order perturbation terms $\Delta M_l^{(1)}$ and $\Delta M_\nu^{(1)}$ should be $S_{2L}\times S_{2R}$ invariant in our model. According to the different group elements of the $S_{2L}\times S_{2R}$ symmetry, three different breaking patterns are investigated in detail, which can be classified as

$$\text{Scenario I}:\quad S_{3L}\times S_{3R}\to S_{2L}^{(23)}\times S_{2R}^{(23)}\to\emptyset,\quad \text{with } S_2^{(23)}=\{S^{(123)},S^{(132)}\},\qquad(11)$$

$$\text{Scenario II}:\quad S_{3L}\times S_{3R}\to S_{2L}^{(13)}\times S_{2R}^{(13)}\to\emptyset,\quad \text{with } S_2^{(13)}=\{S^{(123)},S^{(321)}\},\qquad(12)$$

$$\text{Scenario III}:\quad S_{3L}\times S_{3R}\to S_{2L}^{(12)}\times S_{2R}^{(12)}\to\emptyset,\quad \text{with } S_2^{(12)}=\{S^{(123)},S^{(213)}\}.\qquad(13)$$

As there is no residual symmetry left in the charged-lepton and neutrino sector, the second-order perturbation terms $\Delta M_l^{(2)}$ and $\Delta M_\nu^{(2)}$ are usually arbitrary. For simplicity, $\Delta M_l^{(2)}$ and $\Delta M_\nu^{(2)}$ are chosen to be diagonal in our later discussions.

A. Scenario I ($S_{3L}\times S_{3R}\to S_{2L}^{(23)}\times S_{2R}^{(23)}\to\emptyset$)

For the symmetry-breaking chain $S_{3L}\times S_{3R}\to S_{2L}^{(23)}\times S_{2R}^{(23)}\to\emptyset$, the general form of the first-order perturbation term $\Delta M_l^{(1)}$ can be given by

$$\Delta M_l^{(1)}=\frac{c_l}{3}\begin{pmatrix}\delta_l&0&0\\0&\delta_l&\delta_l\\0&\delta_l&\delta_l\end{pmatrix},\qquad(14)$$

where δ_l is a small real parameter, and $|\delta_l|\ll 1$. Since the residual $S_{2L}\times S_{2R}$ symmetry is eventually broken, there is no remaining symmetry in the charged-lepton mass matrix, and the perturbations in the second-order term $\Delta M_l^{(2)}$ are typically random. However, at least from the perspective of model building, it is more natural to consider simple forms. For simplicity, the following diagonal form is selected:

$$\Delta M_l^{(2)} = \frac{c_l}{3}\begin{pmatrix} -\mathrm{i}\epsilon_l & 0 & 0 \\ 0 & \mathrm{i}\epsilon_l & 0 \\ 0 & 0 & \varepsilon_l \end{pmatrix}. \tag{15}$$

Here, ε_l and ϵ_l are both small perturbative parameters. As the symmetry-breaking terms in the charged-lepton sector are responsible for the generation of muon and electron masses, $|\delta_l|,|\epsilon_l| \ll |\varepsilon_l| < 1$ are required. It is worthwhile to mention that, here, $\Delta M_l^{(2)}$ is assumed to be a complex matrix, which is intentional since we expect appropriate CP violation in the lepton sector.

Now, the mass matrix of the charged-lepton in Eq. (9) can be written as

$$\begin{aligned} M_l &= \frac{c_l}{3}\left[\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} \delta_l & 0 & 0 \\ 0 & \delta_l & \delta_l \\ 0 & \delta_l & \delta_l \end{pmatrix} + \begin{pmatrix} -\mathrm{i}\epsilon_l & 0 & 0 \\ 0 & \mathrm{i}\epsilon_l & 0 \\ 0 & 0 & \varepsilon_l \end{pmatrix}\right] \\ &= \frac{c_l}{3}\begin{pmatrix} 1+\delta_l-\mathrm{i}\epsilon_l & 1 & 1 \\ 1 & 1+\delta_l+\mathrm{i}\epsilon_l & 1+\delta_l \\ 1 & 1+\delta_l & 1+\delta_l+\varepsilon_l \end{pmatrix}, \end{aligned} \tag{16}$$

where $c_l > 0$ and $|\delta_l|,|\epsilon_l| \ll |\varepsilon_l| < 1$. Note that, here, M_l is a complex symmetric matrix and can usually be diagonalized by a unitary matrix. After diagonalizing M_l through $V_l^\dagger M_l V_l^* = \mathrm{Diag}\{m_e, m_\mu, m_\tau\}$, the masses of the three charged-leptons can be obtained as

$$m_\tau \approx c_l\left(1+\frac{\varepsilon_l}{9}+\frac{5\delta_l}{9}\right),\; m_\mu \approx c_l\left(\frac{2\varepsilon_l}{9}+\frac{\delta_l}{9}\right),\; m_e \approx c_l\left(\frac{\delta_l}{3}+\frac{\epsilon_l^2}{6\varepsilon_l}\right). \tag{17}$$

Furthermore, the unitary matrix V_l is found to be

$$\begin{aligned} V_l \approx\; & \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{3} & 1 & \sqrt{2} \\ 0 & -2 & \sqrt{2} \end{pmatrix} + \frac{\mathrm{i}\epsilon_l}{2\sqrt{2}\varepsilon_l}\begin{pmatrix} -1 & -\sqrt{3} & 0 \\ -1 & \sqrt{3} & 0 \\ 2 & 0 & 0 \end{pmatrix} \\ & + \frac{\varepsilon_l}{9\sqrt{3}}\begin{pmatrix} 0 & \sqrt{2} & -1 \\ 0 & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \end{pmatrix}. \end{aligned} \tag{18}$$

It is worth mentioning that the first term in V_l is just the orthogonal matrix V_D that is used to diagonalize the $S_{3L}\times S_{3R}$ symmetry-limit terms $M_l^{(0)}$ and $M_\nu^{(0)}$, as shown in Eq. (8).

In the neutrino sector, the first-order perturbation term $\Delta M_\nu^{(1)}$ with residual $S_{2L}^{(23)}\times S_{2R}^{(23)}$ symmetry can be expressed as

$$\Delta M_\nu^{(1)} = c_\nu\begin{pmatrix} \delta_\nu & 0 & 0 \\ 0 & 0 & \delta_\nu \\ 0 & \delta_\nu & 0 \end{pmatrix}, \tag{19}$$

where $|\delta_\nu| \ll 1$. As in the charged-lepton sector, no residual symmetry remains in the neutrino sector, which allows the second-order perturbation term $\Delta M_\nu^{(2)}$ to be arbitrary. It is worth mentioning that, as the neutrinos are assumed to be Majorana particles, the neutrino mass matrix should be symmetric. For simplicity, $\Delta M_\nu^{(2)}$ is assumed to take the following diagonal form:

$$\Delta M_\nu^{(2)} = c_\nu\begin{pmatrix} -\epsilon_\nu & 0 & 0 \\ 0 & \epsilon_\nu & 0 \\ 0 & 0 & \varepsilon_\nu \end{pmatrix}. \tag{20}$$

Since the symmetry-breaking terms in the neutrino sector are responsible for the breaking of neutrino mass degeneracy, the perturbation parameters should satisfy $|\delta_\nu|,|\epsilon_\nu| \ll |\varepsilon_\nu| < 1$. The general form of the neutrino mass matrix M_ν with perturbations can then be read as

$$\begin{aligned} M_\nu &= c_\nu\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_\nu\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right. \\ &\quad \left. + \begin{pmatrix} \delta_\nu & 0 & 0 \\ 0 & 0 & \delta_\nu \\ 0 & \delta_\nu & 0 \end{pmatrix} + \begin{pmatrix} -\epsilon_\nu & 0 & 0 \\ 0 & \epsilon_\nu & 0 \\ 0 & 0 & \varepsilon_\nu \end{pmatrix}\right] \\ &= c_\nu\begin{pmatrix} 1+r_\nu+\delta_\nu-\epsilon_\nu & r_\nu & r_\nu \\ r_\nu & 1+r_\nu+\epsilon_\nu & r_\nu+\delta_\nu \\ r_\nu & r_\nu+\delta_\nu & 1+r_\nu+\varepsilon_\nu \end{pmatrix}, \end{aligned} \tag{21}$$

where $c_\nu > 0$ and $|r_\nu|,|\delta_\nu|,|\epsilon_\nu| \ll |\varepsilon_\nu| < 1$ are implied. Note that the neutrino mass matrix M_ν derived in this case is the same as that given in Ref. [53]. After diagonalizing M_ν via $V_\nu^\dagger M_\nu V_\nu^* = \mathrm{Diag}\{m_1, m_2, m_3\}$, the three neutrino mass eigenvalues can be given by

$$\begin{aligned} m_3 &\approx c_\nu(1+r_\nu+\varepsilon_\nu), \\ m_2 &\approx c_\nu\left(1+r_\nu+\frac{1}{2}\delta_\nu+\frac{1}{2}\sqrt{(2\epsilon_\nu-\delta_\nu)^2+4r_\nu^2}\right), \\ m_1 &\approx c_\nu\left(1+r_\nu+\frac{1}{2}\delta_\nu-\frac{1}{2}\sqrt{(2\epsilon_\nu-\delta_\nu)^2+4r_\nu^2}\right). \end{aligned} \tag{22}$$

Evidently, the neutrino mass spectrum in Eq. (22) shows a normal mass hierarchy ($m_1 < m_2 < m_3$). Approximately, the unitary matrix V_ν can be expressed as

$$V_\nu \approx \frac{1}{\varepsilon_\nu}\begin{pmatrix} \varepsilon_\nu c_\theta & \varepsilon_\nu s_\theta & r_\nu \\ -\varepsilon_\nu s_\theta & \varepsilon_\nu c_\theta & r_\nu+\delta_\nu \\ (r_\nu+\delta_\nu)s_\theta - r_\nu c_\theta & -(r_\nu+\delta_\nu)c_\theta - r_\nu s_\theta & \varepsilon_\nu \end{pmatrix}, \tag{23}$$

where $c_\theta \equiv \cos\theta$ and $s_\theta \equiv \sin\theta$, with $\tan 2\theta = 2r_\nu/(2\epsilon_\nu - \delta_\nu)$.

The lepton mixing matrix arises from the mismatch between the diagonalization of the charged-lepton mass matrix M_l and that of the neutrino mass matrix M_ν and can be defined as $V_{\rm PMNS} = V_l^\dagger V_\nu$. According to Eqs. (18) and (23), the lepton mixing matrix $V_{\rm PMNS}$ can then be approximately expressed as

$$\begin{aligned} V_{\rm PMNS} = {} & \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3}(c_\theta+s_\theta) & \sqrt{3}(s_\theta-c_\theta) & 0 \\ c_\theta-s_\theta & c_\theta+s_\theta & -2 \\ \sqrt{2}(c_\theta-s_\theta) & \sqrt{2}(c_\theta+s_\theta) & \sqrt{2} \end{pmatrix} \\ & + \frac{{\rm i}\epsilon_l}{2\sqrt{2}\varepsilon_l}\begin{pmatrix} c_\theta-s_\theta & (c_\theta+s_\theta) & -2 \\ \sqrt{3}(c_\theta+s_\theta) & \sqrt{3}(s_\theta-c_\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & + \frac{\varepsilon_l}{9\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}(c_\theta-s_\theta) & \sqrt{2}(c_\theta+s_\theta) & \sqrt{2} \\ s_\theta-c_\theta & -(c_\theta+s_\theta) & 2 \end{pmatrix} \\ & + \frac{r_\nu}{\sqrt{6}\varepsilon_\nu}\begin{pmatrix} 0 & 0 & 0 \\ 2(c_\theta-s_\theta) & 2(c_\theta+s_\theta) & 2 \\ \sqrt{2}(s_\theta-c_\theta) & -\sqrt{2}(c_\theta+s_\theta) & 2\sqrt{2} \end{pmatrix} \\ & + \frac{\delta_\nu}{\sqrt{6}\varepsilon_\nu}\begin{pmatrix} 0 & 0 & -\sqrt{3} \\ -2s_\theta & 2c_\theta & 1 \\ \sqrt{2}s_\theta & -\sqrt{2}c_\theta & \sqrt{2} \end{pmatrix}. \end{aligned} \tag{24}$$

B. Scenario II ($S_{3L}\times S_{3R} \to S_{2L}^{(13)}\times S_{2R}^{(13)} \to \emptyset$)

In case the symmetry-breaking chain is taken as $S_{3L}\times S_{3R} \to S_{2L}^{(13)}\times S_{2R}^{(13)} \to \emptyset$, the first-order perturbation term $\Delta M_l^{(1)}$ can be obtained as

$$\Delta M_l^{(1)} = \frac{c_l}{3}\begin{pmatrix} \delta_l & 0 & \delta_l \\ 0 & \delta_l & 0 \\ \delta_l & 0 & \delta_l \end{pmatrix}, \tag{25}$$

with $|\delta_l| \ll 1$. The second-order perturbation term $\Delta M_l^{(2)}$ is also assumed to be diagonal and takes the same form as in Eq. (15). Finally, the charged-lepton mass matrix M_l can be explicitly expressed as

$$\begin{aligned} M_l = {} & \frac{c_l}{3}\left[\begin{pmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{pmatrix} + \begin{pmatrix} \delta_l & 0 & \delta_l \\ 0 & \delta_l & 0 \\ \delta_l & 0 & \delta_l \end{pmatrix} + \begin{pmatrix} -{\rm i}\epsilon_l & 0 & 0 \\ 0 & {\rm i}\epsilon_l & 0 \\ 0 & 0 & \varepsilon_l \end{pmatrix}\right] \\ = {} & \frac{c_l}{3}\begin{pmatrix} 1+\delta_l-{\rm i}\epsilon_l & 1 & 1+\delta_l \\ 1 & 1+\delta_l+{\rm i}\epsilon_l & 1 \\ 1+\delta_l & 1 & 1+\delta_l+\varepsilon_l \end{pmatrix}, \end{aligned} \tag{26}$$

where $c_l > 0$ and $|\delta_l|, |\epsilon_l| \ll |\varepsilon_l| < 1$ are assumed. The mass matrix M_l of charged-lepton obtained in Eq. (26) is also complex symmetric and can be diagonalized by a unitary matrix. Through a simple perturbation calculation, the masses of the three charged-leptons are approximately given by

$$m_\tau \approx c_l\left(1+\frac{\varepsilon_l}{9}+\frac{5\delta_l}{9}\right),\ m_\mu \approx c_l\left(\frac{2\varepsilon_l}{9}+\frac{\delta_l}{9}\right),\ m_e \approx c_l\left(\frac{\delta_l}{3}+\frac{\epsilon_l^2}{6\varepsilon_l}\right). \tag{27}$$

Note that the charged-lepton mass spectrum obtained here is exactly the same as that given in Eq. (17). The corresponding unitary matrix V_l can be expressed as

$$\begin{aligned} V_l \approx {} & \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{3} & 1 & \sqrt{2} \\ 0 & -2 & \sqrt{2} \end{pmatrix} + \frac{{\rm i}\epsilon_l}{2\sqrt{2}\varepsilon_l}\begin{pmatrix} -1 & -\sqrt{3} & 0 \\ -1 & \sqrt{3} & 0 \\ 2 & 0 & 0 \end{pmatrix} \\ & + \frac{\varepsilon_l}{9\sqrt{3}}\begin{pmatrix} 0 & \sqrt{2} & -1 \\ 0 & \sqrt{2} & -1 \\ 0 & \sqrt{2} & 2 \end{pmatrix}. \end{aligned} \tag{28}$$

For the neutrino sector, the first-order perturbation term $\Delta M_\nu^{(1)}$ in the limit of the residual $S_{2L}^{(13)}\times S_{2R}^{(13)}$ symmetry can be given by

$$\Delta M_\nu^{(1)} = c_\nu\begin{pmatrix} 0 & 0 & \delta_\nu \\ 0 & \delta_\nu & 0 \\ \delta_\nu & 0 & 0 \end{pmatrix}, \tag{29}$$

with $|\delta_\nu| \ll 1$. As given in Eq. (20), the second-order perturbation term $\Delta M_\nu^{(2)}$ is also chosen to be diagonal for simplicity. The mass matrix M_ν of neutrino can now be written as

$$M_\nu = c_\nu\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+r_\nu\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}+\begin{pmatrix}0&0&\delta_\nu\\0&\delta_\nu&0\\\delta_\nu&0&0\end{pmatrix}+\begin{pmatrix}-\epsilon_\nu&0&0\\0&\epsilon_\nu&0\\0&0&\varepsilon_\nu\end{pmatrix}\right]$$
$$= c_\nu\begin{pmatrix}1+r_\nu-\epsilon_\nu & r_\nu & r_\nu+\delta_\nu\\ r_\nu & 1+r_\nu+\delta_\nu+\epsilon_\nu & r_\nu\\ r_\nu+\delta_\nu & r_\nu & 1+r_\nu+\varepsilon_\nu\end{pmatrix}, \quad (30)$$

where $c_\nu>0$ and $|r_\nu|,|\delta_\nu|,|\epsilon_\nu| \ll |\varepsilon_\nu| < 1$. By diagonalizing the above matrix, we obtain the mass eigenvalues of three neutrinos with high accuracy

$$m_3 \approx c_\nu(1+r_\nu+\varepsilon_\nu)\,,$$
$$m_2 \approx c_\nu\left(1+r_\nu+\frac{1}{2}\delta_\nu+\frac{1}{2}\sqrt{(2\epsilon_\nu+\delta_\nu)^2+4r_\nu^2}\right)\,,$$
$$m_1 \approx c_\nu\left(1+r_\nu+\frac{1}{2}\delta_\nu-\frac{1}{2}\sqrt{(2\epsilon_\nu+\delta_\nu)^2+4r_\nu^2}\right)\,. \quad (31)$$

The unitary matrix used to diagonalize M_ν via $V_\nu^\dagger M_\nu V_\nu^* = \mathrm{Diag}\{m_1,m_2,m_3\}$ is found to be

$$V_\nu \approx \frac{1}{\varepsilon_\nu}\begin{pmatrix}\varepsilon_\nu c_\theta & \varepsilon_\nu s_\theta & r_\nu+\delta_\nu\\ -\varepsilon_\nu s_\theta & \varepsilon_\nu c_\theta & r_\nu\\ r_\nu s_\theta-(r_\nu+\delta_\nu)c_\theta & -(r_\nu+\delta_\nu)s_\theta - r_\nu c_\theta & \varepsilon_\nu\end{pmatrix}, \quad (32)$$

with $\tan 2\theta = 2r_\nu/(2\epsilon_\nu+\delta_\nu)$.

In this case, the explicit form of the lepton mixing matrix $V_{\rm PMNS} = V_l^\dagger V_\nu$ can be approximately derived as

$$V_{\rm PMNS} = \frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{3}(c_\theta+s_\theta) & \sqrt{3}(s_\theta-c_\theta) & 0\\ c_\theta-s_\theta & c_\theta+s_\theta & -2\\ \sqrt{2}(c_\theta-s_\theta) & \sqrt{2}(c_\theta+s_\theta) & \sqrt{2}\end{pmatrix}$$
$$+\frac{{\rm i}\epsilon_l}{2\sqrt{2}\varepsilon_l}\begin{pmatrix}c_\theta-s_\theta & (c_\theta+s_\theta) & -2\\ \sqrt{3}(c_\theta+s_\theta) & \sqrt{3}(s_\theta-c_\theta) & 0\\ 0&0&0\end{pmatrix}$$
$$+\frac{\varepsilon_l}{9\sqrt{3}}\begin{pmatrix}0&0&0\\ \sqrt{2}(c_\theta-s_\theta) & \sqrt{2}(c_\theta+s_\theta) & \sqrt{2}\\ s_\theta-c_\theta & -(c_\theta+s_\theta) & 2\end{pmatrix}$$
$$+\frac{r_\nu}{\sqrt{6}\varepsilon_\nu}\begin{pmatrix}0&0&0\\ 2(c_\theta-s_\theta) & 2(c_\theta+s_\theta) & 2\\ \sqrt{2}(s_\theta-c_\theta) & -\sqrt{2}(c_\theta+s_\theta) & 2\sqrt{2}\end{pmatrix}$$
$$+\frac{\delta_\nu}{\sqrt{6}\varepsilon_\nu}\begin{pmatrix}0&0&\sqrt{3}\\ 2c_\theta & 2s_\theta & 1\\ -\sqrt{2}c_\theta & -\sqrt{2}s_\theta & \sqrt{2}\end{pmatrix}. \quad (33)$$

C. Scenario III ($S_{3L}\times S_{3R}\to S_{2L}^{(12)}\times S_{2R}^{(12)}\to\emptyset$)

When the symmetry-breaking chain is assumed to be $S_{3L}\times S_{3R}\to S_{2L}^{(12)}\times S_{2R}^{(12)}\to\emptyset$, the first-order perturbation term $\Delta M_l^{(1)}$ is found to be

$$\Delta M_l^{(1)} = \frac{c_l}{3}\begin{pmatrix}\delta_l&\delta_l&0\\ \delta_l&\delta_l&0\\ 0&0&\delta_l\end{pmatrix}, \quad (34)$$

with $|\delta_l|\ll 1$. The second-order perturbation term $\Delta M_l^{(2)}$ is also set as a diagonal matrix, as expressed in Eq. (15). According to the above discussions, the mass matrix M_l of the charged-lepton can be expressed as

$$M_l = \frac{c_l}{3}\left[\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}+\begin{pmatrix}\delta_l&\delta_l&0\\ \delta_l&\delta_l&0\\ 0&0&\delta_l\end{pmatrix}+\begin{pmatrix}-{\rm i}\epsilon_l&0&0\\0&{\rm i}\epsilon_l&0\\0&0&\varepsilon_l\end{pmatrix}\right]$$
$$= \frac{c_l}{3}\begin{pmatrix}1+\delta_l-{\rm i}\epsilon_l & 1+\delta_l & 1\\ 1+\delta_l & 1+\delta_l+{\rm i}\epsilon_l & 1\\ 1&1&1+\delta_l+\varepsilon_l\end{pmatrix}, \quad (35)$$

where $c_l>0$ and $|\delta_l|,|\epsilon_l|\ll|\varepsilon_l|<1$. By diagonalizing the charged-lepton mass matrix M_l, we obtain the masses of three charged-leptons:

$$m_\tau \approx c_l\left(1+\frac{\varepsilon_l}{9}+\frac{5\delta_l}{9}\right), m_\mu \approx c_l\left(\frac{2\varepsilon_l}{9}+\frac{4\delta_l}{9}\right),$$
$$m_e \approx c_l\left(\frac{\epsilon_l^2}{6\varepsilon_l}-\frac{\delta_l\epsilon_l^2}{3\varepsilon_l^2}\right). \quad (36)$$

The unitary matrix V_l used to diagonalize M_l is given by

$$V_l \approx \frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{3}&1&\sqrt{2}\\ -\sqrt{3}&1&\sqrt{2}\\ 0&-2&\sqrt{2}\end{pmatrix}+\frac{{\rm i}\epsilon_l}{2\sqrt{2}\varepsilon_l}\begin{pmatrix}-1&-\sqrt{3}&0\\-1&\sqrt{3}&0\\2&0&0\end{pmatrix}$$
$$+\frac{\varepsilon_l}{9\sqrt{3}}\begin{pmatrix}0&\sqrt{2}&-1\\0&\sqrt{2}&-1\\0&\sqrt{2}&2\end{pmatrix}. \quad (37)$$

For the neutrino sector, the first-order perturbation term $\Delta M_\nu^{(1)}$ constrained by the residual $S_{2L}^{(12)}\times S_{2R}^{(12)}$ symmetry is found to be

$$\Delta M_\nu^{(1)} = c_\nu\begin{pmatrix}0&\delta_\nu&0\\ \delta_\nu&0&0\\ 0&0&\delta_\nu\end{pmatrix}, \quad (38)$$

with $|\delta_\nu|\ll 1$. The second-order perturbation term $\Delta M_\nu^{(2)}$ is set to the same form as in Eq. (20). The mass matrix M_ν of the neutrino can then be expressed as

$$\begin{aligned} M_\nu &= c_\nu\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+r_\nu\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\right.\\ &\left.+\begin{pmatrix}0&\delta_\nu&0\\\delta_\nu&0&0\\0&0&\delta_\nu\end{pmatrix}+\begin{pmatrix}-\epsilon_\nu&0&0\\0&\epsilon_\nu&0\\0&0&\varepsilon_\nu\end{pmatrix}\right]\\ &= c_\nu\begin{pmatrix}1+r_\nu-\epsilon_\nu & r_\nu+\delta_\nu & r_\nu\\ r_\nu+\delta_\nu & 1+r_\nu+\epsilon_\nu & r_\nu\\ r_\nu & r_\nu & 1+r_\nu+\delta_\nu+\varepsilon_\nu\end{pmatrix}, \end{aligned} \tag{39}$$

where $c_\nu>0$ and $|r_\nu|,|\delta_\nu|,|\epsilon_\nu|\ll|\varepsilon_\nu|<1$. Diagonalizing the above matrix M_ν, the three neutrino mass eigenvalues can be derived as

$$\begin{aligned} m_3 &\approx c_\nu(1+\delta_\nu+r_\nu+\varepsilon_\nu),\\ m_2 &\approx c_\nu\left(1+r_\nu+\sqrt{(\delta_\nu+r_\nu)^2+\epsilon_\nu^2}\right),\\ m_1 &\approx c_\nu\left(1+r_\nu-\sqrt{(\delta_\nu+r_\nu)^2+\epsilon_\nu^2}\right). \end{aligned} \tag{40}$$

The unitary matrix used to diagonalize the neutrino mass matrix can be approximately given by

$$V_\nu \approx \frac{1}{\varepsilon_\nu}\begin{pmatrix}\varepsilon_\nu c_\theta & \varepsilon_\nu s_\theta & r_\nu\\ -\varepsilon_\nu s_\theta & \varepsilon_\nu c_\theta & r_\nu\\ r_\nu(s_\theta-c_\theta) & -r_\nu(c_\theta+s_\theta) & \varepsilon_\nu\end{pmatrix}, \tag{41}$$

with $\tan 2\theta=(r_\nu+\delta_\nu)/\epsilon_\nu$. From Eqs. (37) and (41), we can derive the lepton mixing matrix $V_{\rm PMNS}=V_l^\dagger V_\nu$. More explicitly,

$$\begin{aligned} V_{\rm PMNS} &= \frac{1}{\sqrt6}\begin{pmatrix}\sqrt3(c_\theta+s_\theta) & \sqrt3(s_\theta-c_\theta) & 0\\ c_\theta-s_\theta & c_\theta+s_\theta & -2\\ \sqrt2(c_\theta-s_\theta) & \sqrt2(c_\theta+s_\theta) & \sqrt2\end{pmatrix}\\ &+\frac{{\rm i}\epsilon_l}{2\sqrt2\varepsilon_l}\begin{pmatrix}c_\theta-s_\theta & (c_\theta+s_\theta) & -2\\ \sqrt3(c_\theta+s_\theta) & \sqrt3(s_\theta-c_\theta) & 0\\ 0&0&0\end{pmatrix}\\ &+\frac{\varepsilon_l}{9\sqrt3}\begin{pmatrix}0&0&0\\ \sqrt2(c_\theta-s_\theta) & \sqrt2(c_\theta+s_\theta) & \sqrt2\\ s_\theta-c_\theta & -(c_\theta+s_\theta) & 2\end{pmatrix}\\ &+\frac{r_\nu}{\sqrt6\varepsilon_\nu}\begin{pmatrix}0&0&0\\ 2(c_\theta-s_\theta) & 2(c_\theta+s_\theta) & 2\\ \sqrt2(s_\theta-c_\theta) & -\sqrt2(c_\theta+s_\theta) & 2\sqrt2\end{pmatrix}. \end{aligned} \tag{42}$$

At this stage, we have derived the lepton mixing matrix corresponding to the three distinct symmetry-breaking chains. In these scenarios, various symmetry-breaking patterns yield different parameterizations of the lepton mass matrices, which ultimately lead to distinct lepton mixing matrices. By comparing the obtained lepton mixing matrix with its standard parametrization, one can easily extract the three neutrino mixing angles and the CP-violating phase, which we elaborate on in the next section.

IV. NUMERICAL ANALYSIS AND PREDICTIONS FOR CP VIOLATION

In general, the lepton mixing matrix $V_{\rm PMNS}$ can be parameterized by three mixing angles (θ_{12}, θ_{13}, and θ_{23}) and one Dirac CP-violating phase δ. If neutrinos are Majorana particles, $V_{\rm PMNS}$ also includes two additional Majorana phases ρ and σ. The standard parameterization of $V_{\rm PMNS}$ can be expressed as [69]

$$V_{\rm PMNS}=\begin{pmatrix}c_{13}c_{12} & c_{13}s_{12} & s_{13}{\rm e}^{-{\rm i}\delta}\\ -s_{12}c_{23}-c_{12}s_{13}s_{23}{\rm e}^{{\rm i}\delta} & c_{12}c_{23}-s_{12}s_{13}s_{23}{\rm e}^{{\rm i}\delta} & c_{13}s_{23}\\ s_{12}s_{23}-c_{12}s_{13}c_{23}{\rm e}^{{\rm i}\delta} & -c_{12}s_{23}-s_{12}s_{13}c_{23}{\rm e}^{{\rm i}\delta} & c_{13}c_{23}\end{pmatrix}\cdot P_{\rm M}, \tag{43}$$

where $c_{ij}\equiv\cos\theta_{ij}$, $s_{ij}\equiv\sin\theta_{ij}$, and $P_{\rm M}={\rm Diag}\{{\rm e}^{{\rm i}\rho},{\rm e}^{{\rm i}\sigma},1\}$. Based on the previous discussions, it is evident that there are nine parameters in our model, that is, four parameters $(c_l,\delta_l,\epsilon_l,\varepsilon_l)$ in the charged-lepton sector and five parameters $(c_\nu,r_\nu,\delta_\nu,\epsilon_\nu,\varepsilon_\nu)$ in the neutrino sector. These parameters can be fully determined by nine corresponding observables, namely, three charged-lepton masses (m_e,m_μ,m_τ), three neutrino masses (m_1,m_2,m_3), and three neutrino mixing angles $(\theta_{12},\theta_{13},\theta_{23})$. On this basis, we can further obtain the predicted value of the CP-violating phase. For convenience, the latest best-fit values of neutrino oscillation parameters for the normal neutrino mass hierarchy provided by NuFIT 6.0 (2024) [14] are adopted in the following numerical analysis.

For Scenario I, the two neutrino mass-squared differences can be calculated from Eq. (22):

$$\Delta m_{31}^2\approx c_\nu^2\varepsilon_\nu(2+\varepsilon_\nu)\,,\quad \Delta m_{21}^2\approx 2c_\nu^2\sqrt{(2\epsilon_\nu-\delta_\nu)^2+4r_\nu^2}\,. \tag{44}$$

The three neutrino masses in Eq. (22) are nearly degenerate, indicating that the effective neutrino masses in tritium beta decays and neutrinoless double-beta decays are of the same order as the neutrino mass scale parameter c_ν. In the conservative case, $c_\nu\approx 0.03$ eV is taken, which is consistent with the current cosmological observations [70]. By using the best-fit value of $\Delta m_{31}^2=2.534\times 10^{-3}$ eV^2, one can obtain

$$\varepsilon_\nu(2+\varepsilon_\nu) \approx \frac{\Delta m_{31}^2}{c_\nu^2} \to \varepsilon_\nu \approx 0.953 \,. \tag{45}$$

By comparing the lepton mixing matrix $V_{\rm PMNS}$ in Eq. (24) with the standard parametrization in Eq. (43), one can find that

$$\sin^2\theta_{12} = \frac{|V_{e2}|^2}{1-|V_{e3}|^2} \approx \frac{1}{2}(1-\sin 2\theta) \,. \tag{46}$$

Utilizing the best-fit value of $\theta_{12} \approx 33.68°$, we obtain $\theta = 11.32°$. With the help of Eq. (44) and the relation $\tan 2\theta = 2r_\nu/(2\epsilon_\nu - \delta_\nu)$ defined before, we can obtain

$$\frac{2r_\nu}{\varepsilon_\nu(2+\varepsilon_\nu)} = \cos\theta\sin\theta\frac{\Delta m_{21}^2}{\Delta m_{31}^2} \,. \tag{47}$$

By inputting the best-fit value of $\Delta m_{21}^2 = 7.49\times10^{-5}$ eV^2, one can then derive

$$r_\nu \approx 8.01\times10^{-3} \,. \tag{48}$$

Adopting the standard parametrization in Eq. (43), we can also obtain

$$\sin^2\theta_{23} \approx \frac{|V_{\mu3}|^2}{1-|V_{e3}|^2} \approx \frac{2}{3}\left(1-\frac{2\varepsilon_l}{9}-\frac{2r_\nu}{\varepsilon_\nu}-\frac{\delta_\nu}{\varepsilon_\nu}\right) ,$$
$$\sin^2\theta_{13} \approx |V_{e3}|^2 \approx \frac{1}{2}\left(\frac{\delta_\nu^2}{\varepsilon_\nu^2}+\frac{\epsilon_l^2}{\varepsilon_l^2}\right) . \tag{49}$$

Combining Eqs. (17) and (49), one finally obtains

$$\varepsilon_l \approx 0.275,\ \epsilon_l \approx -5.31\times10^{-2},\ \delta_l \approx -4.27\times10^{-3},\ \delta_\nu \approx 7.69\times10^{-2}, \tag{50}$$

by inputting the best-fit value of $\theta_{23} = 48.5°$, $\theta_{13} = 8.52°$, together with the charged-lepton masses $m_e = 0.488$ MeV, $m_\mu = 102.877$ MeV and $m_\tau = 1747.43$ MeV at the electroweak scale [71]. The magnitude of ϵ_ν can be obtained through

$$\epsilon_\nu = \frac{1}{2}\left(\frac{2r_\nu}{\tan 2\theta}+\delta_\nu\right) \approx 5.76\times10^{-2} \,. \tag{51}$$

Through the above calculations, it is easy to find that all the model parameters are in good agreement with our initial expectations that $|\delta_l|, |\epsilon_l| \ll |\varepsilon_l| < 1$ and $|r_\nu|, |\delta_\nu|, |\epsilon_\nu| \ll |\varepsilon_\nu| < 1$.

The Dirac CP-violating phase δ can be extracted from the Jarlskog invariant [72, 73]

$$\mathcal{J} = {\rm Im}[V_{\mu1}V_{\tau2}V_{\mu2}^*V_{\tau1}^*] = \frac{1}{8}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\sin\delta \,. \tag{52}$$

It can be found that the predicted value of δ is approximately

$$\delta \approx 294.6° \,, \tag{53}$$

which falls within the 3σ interval and will be tested in future neutrino oscillation experiments. Undoubtedly, the numerical prediction of the Dirac CP-violating phase presented here depends on the set-up of the second-order perturbation term in the previous section.

The two Majorana CP-violating phases ρ and σ can also be extracted as

$$\rho = \arctan\left[\frac{1}{2}\frac{\epsilon_l}{\varepsilon_l}\frac{c_\theta - s_\theta}{c_\theta + s_\theta}\right] \approx -3.68°,$$
$$\sigma = \arctan\left[\frac{1}{2}\frac{\epsilon_l}{\varepsilon_l}\frac{c_\theta + s_\theta}{s_\theta - c_\theta}\right] \approx 8.24° \,, \tag{54}$$

which are close to the trivial value.

Finally, we present the sum of neutrino masses $\sum m_\nu$ and the effective neutrino mass $\langle m\rangle_{ee}$ of the neutrinoless double-beta decay. Thus,

$$\sum m_\nu = m_1 + m_2 + m_3 \approx 0.12 \text{ eV} \,,$$
$$\langle m\rangle_{ee} = \left|\sum_i m_i V_{ei}^2\right| \approx 2.91\times10^{-2} \text{ eV} \,. \tag{55}$$

Following the same procedure as that used for Scenario I, we can calculate the values of the model parameters in Scenario II and Scenario III and provide the predicted value of the Dirac CP-violating phase. In Table 1, we list the values of the model parameters and the predicted Dirac CP-violating phase for each scenario. All three scenarios are in good agreement with current experimental data, and the predicted values of the Dirac CP-violating phase all lie within the 3σ range.

To assess the validity of the model more comprehens-

Table 1. Values of the model parameters and the predicted Dirac CP-violating phase for each scenario.

Scenario	Model parameters	Dirac CP-violating phase
Scenario I	$\varepsilon_l \approx 0.275$, $\delta_l \approx -4.27\times10^{-3}$, $\epsilon_l \approx -5.31\times10^{-2}$ $\varepsilon_\nu \approx 0.953$, $r_\nu \approx 8.01\times10^{-3}$, $\delta_\nu \approx 7.69\times10^{-2}$, $\epsilon_\nu \approx 5.76\times10^{-2}$	$\delta \approx 294.6°$
Scenario II	$\varepsilon_l \approx 0.275$, $\delta_l \approx -4.27\times10^{-3}$, $\epsilon_l \approx -5.31\times10^{-2}$ $\varepsilon_\nu \approx 0.953$, $r_\nu \approx 8.01\times10^{-3}$, $\delta_\nu \approx 7.69\times10^{-2}$, $\epsilon_\nu \approx -1.93\times10^{-2}$	$\delta \approx 302.3°$
Scenario III	$\varepsilon_l \approx 0.159$, $\delta_l \approx 5.96\times10^{-2}$, $\epsilon_l \approx -3.34\times10^{-2}$ $\varepsilon_\nu \approx 0.953$, $r_\nu \approx 5.87\times10^{-2}$, $\delta_\nu \approx -6.67\times10^{-2}$, $\epsilon_\nu \approx -1.92\times10^{-2}$	$\delta \approx 287.0°$

ively, the 3σ ranges of three neutrino mixing angles and two mass-squared differences from the latest global analysis of neutrino oscillation data are implemented [14]:

$$\begin{aligned} &\theta_{12} = 31.63^\circ \to 35.95^\circ, \quad \theta_{13} = 8.18^\circ \to 8.87^\circ\,, \\ &\theta_{23} = 41.0^\circ \to 50.5^\circ, \; \Delta m^2_{21} = (6.92 \to 8.05)\times 10^{-5}\mathrm{eV}^2, \\ &\Delta m^2_{31} = (2.463 \to 2.606)\times 10^{-3}\mathrm{eV}^2\,. \end{aligned} \tag{56}$$

For the three scenarios, the corresponding predictions of the Dirac CP-violating phase δ can be given by

$$\begin{aligned} &\delta = 281.2^\circ \to 338.7^\circ \quad \text{for Scenario I}\,, \\ &\delta = 287.0^\circ \to 342.2^\circ \quad \text{for Scenario II}\,, \\ &\delta = 282.7^\circ \to 297.0^\circ \quad \text{for Scenario III}\,, \end{aligned} \tag{57}$$

which all lie within the 3σ range. As examples, the variations of the predicted Dirac CP-violating phase δ with $\sin^2\theta_{23}$ for Scenario I, Scenario II, and Scenario III are shown in Fig. 1.

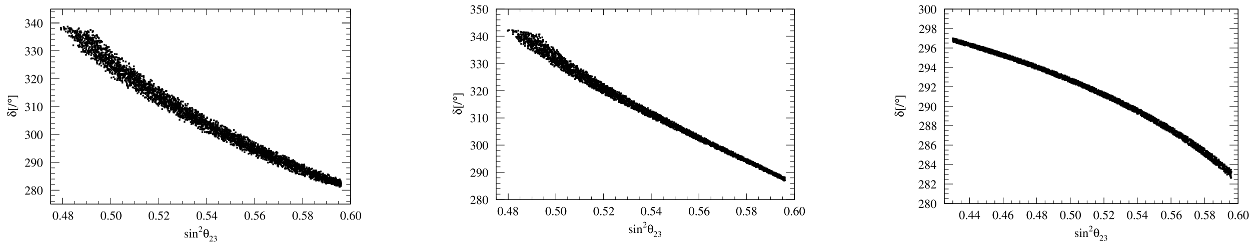

Fig. 1. Predicted Dirac CP-violating phase δ as a function of $\sin^2\theta_{23}$ for (a) Scenario I, (b) Scenario II, and (c) Scenario III.

V. CONCLUSION

Flavor symmetry is one of the main ways to explain lepton mass spectra and flavor mixing. In this study, we adopt a phenomenological approach and apply the $S_{3L}\times S_{3R}$ flavor symmetry to the mass matrices of charged-leptons and neutrinos in a similar way. To explain the realistic lepton mass hierarchy, the symmetry-breaking chain $S_{3L}\times S_{3R} \to S_{2L}\times S_{2R} \to \emptyset$ is introduced to both the charged-lepton and neutrino mass matrices. For three residual subgroups $S^{ij}_{2L}\times S^{ij}_{2R}$ $(ij = 23, 13, 12)$, we systematically analyze the various parameterizations of the lepton mass matrices and derive the corresponding lepton mass spectra, neutrino mixing angles, and Dirac CP-violating phase.

To test the viability of the framework, a detailed numerical analysis is performed. While the three symmetry-breaking chains impose different structural constraints on the mass matrices and lead to distinct lepton mass spectra and mixing matrices, all three scenarios are found to agree well with the current experimental constraints. Specifically, there are nine parameters in the model, that is, four parameters in the charged-lepton sector $(c_l, \delta_l, \epsilon_l, \varepsilon_l)$ and five parameters in the neutrino sector $(c_\nu, r_\nu, \delta_\nu, \epsilon_\nu, \varepsilon_\nu)$. By inputting nine experimental observables, namely three charged-lepton masses (m_e, m_μ, m_τ), three neutrino masses (m_1, m_2, m_3), and three mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$, we can fully determine all model parameters and make predictions for the Dirac CP-violating phase δ. For instance, with the latest best-fit values provided by NuFIT 6.0 (2024), the predicted values of the Dirac CP-violating phase corresponding to the three distinct symmetry-breaking chains are $\delta \approx 294.6^\circ$, 302.3°, and 287.0°. To better assess the viability and generality of the model, we further extend the ranges of the input observables to their full 3σ intervals and perform a comprehensive random scan over the model parameter space. The allowed ranges of δ are $281.2^\circ \to 338.7^\circ$, $287.0^\circ \to 342.2^\circ$, and $282.7^\circ \to 297.0^\circ$, respectively.

Through a comprehensive numerical analysis, it is further substantiated that the symmetry-breaking scheme presented in this paper agrees well with the current experimental data, offering significant theoretical insights into lepton masses, flavor mixing, and CP violation within the context of flavor symmetries. The symmetry-breaking chain $S_{3L}\times S_{3R} \to S_{2L}\times S_{2R} \to \emptyset$ may also be extended to the quark sector, thereby providing valuable information for elucidating the mass spectra and flavor mixing of quarks.

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