Covariant canonical-spinor amplitudes for partial wave analysis*

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Abstract: We propose a covariant orbital-spin (LS) decomposed amplitude for the partial wave analysis using the massive spinor-helicity formalism. First, we review the traditional-LS method in the little group space and the Zemach tensor method in the double cover of the $SO(3)$ space. To recover the $SO(3,1)$ Lorentz covariance, several Lorentz covariant LS tensors have been constructed through different methods: covariant tensor, covariant projection tensor in pure-spin and general-spin schemes. However, performing an intrinsic separation between LS coupling while maintaining covariance is not straightforward. We utilize the massive canonical-spinor variables to determine general three-point amplitudes, where the LS decomposition is realized in a single little group space by projecting little group indices of each particle into one, while ensuring Lorentz covariance by the spinor form naturally. This covariant spinor method allows direct evaluation in any frame and offers a streamlined treatment of cascade decays within a single frame without additional alignment rotations needed in non-covariant approaches. As a benchmark, we implement the method in TF-PWA and analyze $\Lambda_c^+ \to \Lambda\pi^+\pi^0$, finding consistent fit results across the helicity, traditional-LS, and canonical-spinor amplitudes. This validates the canonical-spinor amplitude as a practical tool for modern partial wave analyses of complex decay chains.

Keywords: partial wave analysis, partial wave amplitude, covariant orbital-spin amplitude, canonical-spinor amplitude, cascade decay

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I. INTRODUCTION

In high-energy physics experiments, multi-body final states are among the most common observables. The same final state often does not originate from a single decay chain but is produced through several intermediate resonances together with non-resonant processes. Because different resonances can overlap within invariant mass spectra and each exhibits unique angular structures determined by its spin and orbital angular momentum, relying solely on mass peak shapes is generally insufficient to reliably distinguish between them. Partial Wave Analysis (PWA) provides a systematic framework to address this problem; see Refs. [1−5] for reviews. Its central idea is to construct the total amplitude built from a set of partial wave amplitudes with definite quantum numbers. By exploiting information from invariant mass spectra and angular distributions, PWA can distinguish contributions from different resonances and partial waves, determine their relative magnitudes and phases, and extract information such as fit fraction, interference fractions, and decay asymmetry parameters. Therefore, PWA is an essential tool in multi-body hadronic decays for establishing the presence of resonances and determining spin-parity quantum numbers.

In the PWA of cascade decays, the total amplitude is typically constructed by connecting a sequence of two-body decay amplitudes via propagators of intermediate resonances. Therefore, the method of building the two-body decay amplitude serves as the foundation of the entire analysis. Early schemes for constructing two-body partial wave amplitudes largely relied on a specific refer-

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ence frame. Among the most classic is the helicity formalism introduced by Jacob and Wick [6−8], which defines helicity amplitudes directly in the two-body center-of-mass (COM) frame. It provides a clear physical picture and has been widely developed and extensively applied in studies of particle polarization [2, 3, 9−15].

The closely related method is the traditional orbital-spin (LS) method in canonical formalism. In the two-body COM frame, it expresses the amplitude as the coupling between the orbital part and the spin part. The angular momentum coupling is carried out in the little group representation fixed by the parent standard momentum and is implemented explicitly by the Clebsch-Gordan coefficients (CGCs). This separated structure facilitates the phenomenological inclusion of L-dependent barrier factors [16]. It should be emphasized that the helicity basis and the traditional-LS basis are two choices of spin bases for the same two-particle Hilbert space, related by a unitary change of basis, so they yield identical observables when a complete set of partial waves is retained [17]. However, differences can still arise in practice from different conventions for kinematic and energy-dependent factors, and these effects have been explicitly compared [18, 19].

Besides coupling the orbital and spin parts via CGCs in the little group representation, the Zemach tensor method [20, 21] provides an alternative realization of the orbital-spin coupling. It does not use CGCs explicitly in the little group representation. Instead, one first constructs the angular momentum tensors associated with the orbital and spin parts, and then performs the required coupling at the tensor level using the invariant tensors of $SO(3)$ and $SU(2)$. These invariant tensors are equivalent in effect to the CGCs coupling in the little group representation. In this way, the Zemach tensor method gives an equivalent tensor realization of the LS partial wave coupling.

However, these non-covariant methods share a fundamental drawback: the definition and calculation of the amplitudes are explicitly dependent on the COM frame. When computing any event, the particle four-momenta must be boosted to this specific reference frame. A more subtle yet critical issue arises when the process involves cascade decays with multiple distinct decay chains: the spin states of the same final-state particle in different chains are often defined by different sequences of Lorentz transformations, corresponding to different spin quantization axes and phase conventions. Simply adding the amplitudes from different chains coherently may introduce unphysical convention dependence in the interference terms [22]. To address this issue, the helicity and canonical formalisms differ in their technical treatments [23], but they share the same underlying goal: to bring the spin states of the same physical particle appearing in different decay chains into a common choice of quantization axis and phase convention [22−27]. Only after such an alignment of spin states can the amplitudes from different decay chains be coherently summed in a correct and physically meaningful way.

To achieve full covariance for computational simplicity and physical clarity, methods for constructing covariant two-body partial wave amplitudes have been developed. Chung [1, 28−30] pioneered a systematic scheme by constructing covariant orbital and spin tensors, establishing a direct correspondence between helicity coupling amplitudes and the corresponding covariant tensor structures. Subsequently, Filippini *et al.* [31] further systematically constructed sets of covariant spin tensor bases suitable for spectroscopic analysis, discussing their correspondence and equivalence with the covariant helicity approach. For practical hadron spectroscopy and charmonium decay analyses, Zou and Bugg [32, 33] summarized a widely adopted cookbook style of constructing covariant tensors, referred to as the covariant tensor method. These covariant tensors provide partial wave amplitude expressions for specific decay channels, serving as one of the main PWA schemes adopted by the BESIII Collaboration [34−37]. Subsequent research has continuously refined this framework, forming the now mainstream methods of constructing LS amplitudes using covariant tensors [31−33, 38−49]. Besides these methods, another covariant method starts from a covariant effective Lagrangian and uses the Feynman rules to systematically generate vertices and propagators, thereby organizing the total amplitude for phenomenological analyses of related reactions and decays [50−55].

The methods of constructing LS amplitudes using covariant tensors can essentially be viewed as extending the Zemach tensor method to the Lorentz $SO(3,1)$ representation. Its key idea is to construct a covariant LS coupling structure and then fully contract it with the external spin wave functions over Lorentz indices, thereby obtaining the corresponding partial wave amplitudes. To obtain a general construction of covariant coupling structures, a systematic approach has been developed based on covariant projection tensors [48, 49]. This approach, referred to as the covariant projection tensor method, provides the required covariant LS coupling structures in a fully general form.

In the covariant projection tensor method, this covariant LS coupling structure can be understood as follows. Taking the standard momentum fixed by the parent momentum as the reference, each angular momentum object is first related between the Lorentz representation and the little group representation by means of the rest frame spin wave functions, so that the angular momentum coupling can be carried out within a common little group representation. After the coupling is completed, the result is mapped back to the Lorentz representation using the rest frame spin wave function associated with the total angular momentum. In this way, one obtains the projection tensor carrying only Lorentz indices, which implements the covariant coupling between the orbital and spin parts.

Depending on the choice of external spin wave functions, all the tensor methods of constructing LS amplitudes, including Zemach tensor, covariant tensor, and covariant projection tensor methods, admit two concrete implementations, referred to as the Pure-spin (PS) scheme and the General-spin (GS) scheme, as follows:

1. **Pure-spin scheme (PS scheme)** [28−30, 46, 48, 49]:

$$A(k_3,p_1^*,p_2^*;L,S)=\underbrace{\Gamma(k_3,p_1^*,p_2^*;L,S)}_{LS\ \text{coupling structure}}\times\underbrace{U(k_3,k_1,k_2;s_3,s_1,s_2)}_{\text{pure-spin part}}. \tag{1}$$

In this definition, the COM frame amplitude is organized such that the pure-spin part is a purely intrinsic-spin object. As a result, the pure-spin part carries no explicit dependence on the orbital component, and the LS separation is explicit in the COM frame, with all angular dependence encoded in the orbital structures. By applying a Lorentz boost, this amplitude can be evaluated in any frame, although the COM momenta are still necessarily introduced. Specifically, the amplitude can be written as

$$\begin{aligned}&A(p_3,p_1,p_2;L,S)\\&=\Gamma(p_3,p_1,p_2;L,S)\times U(p_3,p_1,p_2;s_3,s_1,s_2)\\&\quad\times D^{s_1}\left(L(\mathbf{p}_3)L(\mathbf{p}_1^*)L^{-1}(\mathbf{p}_3)\right)D^{s_2}\left(L(\mathbf{p}_3)L(\mathbf{p}_2^*)L^{-1}(\mathbf{p}_3)\right),\end{aligned} \tag{2}$$

where $D^{s_i}\left(L(\mathbf{p}_3)L(\mathbf{p}_i^*)L^{-1}(\mathbf{p}_3)\right)$ $(i=1,2)$ are the Lorentz transformation matrices induced by the boost that takes the amplitude from the COM frame to any other frame. Although it can be evaluated in any reference frame, the associated Lorentz transformation depends on the particle momenta $\mathbf{p}_i^*$ $(i=1,2)$ in the COM frame. Therefore, the PS scheme is not fully manifestly covariant at the level of its formal definition.

2. **General-spin scheme (GS scheme)** [31−33, 38–45, 47, 48]:

$$\begin{aligned}C(p_3,p_1,p_2;L,S)=&\underbrace{\Gamma(p_3,p_1,p_2;L,S)}_{\text{covariant } LS\ \text{coupling structure}}\\&\times\underbrace{U(p_3,p_1,p_2;s_3,s_1,s_2)}_{\text{general-spin part}}.\end{aligned} \tag{3}$$

The resulting expressions are manifestly covariant in the sense that they can be evaluated directly in any frame using the corresponding four-momenta. In this definition, the general-spin part is constructed directly from general-spin structures evaluated with the momenta

$$\begin{aligned}U(p_3,p_1,p_2;s_3,s_1,s_2)=&\ u(p_3;s_3)\bar{u}(p_1;s_1)\bar{u}(p_2;s_2)\\&\xrightarrow{\text{COM}} u(k_3;s_3)\bar{u}(p_1^*;s_1)\bar{u}(p_2^*;s_2).\end{aligned} \tag{4}$$

Note that in the COM frame, the general-spin part depends on the relative momentum direction and thus contains residual contributions from the orbital part. Consequently, the separation between L and S does not follow the traditional-LS method, since the general-spin part here is not purely intrinsic-spin objects. Only in the non-relativistic limit, the general-spin part is reduced to $u(k_3;s_3)\bar{u}(k_1;s_1)\bar{u}(k_2;s_2)$, which is the pure spin part $U(k_3,k_1,k_2;s_3,s_1,s_2)$ in Eq. (2), and the covariant amplitude would have clear LS separation.

Therefore, it can be seen that neither scheme simultaneously provides a manifestly covariant construction that avoids boosts to the two-body COM frame and an LS separation that is identical to the traditional-LS definition. Moreover, when massless gauge bosons are involved, formulations based on polarization vectors, or analogous wave functions, typically require gauge invariance to be enforced explicitly. In practice, one must select a complete set of gauge-invariant and linearly independent vertex structures to eliminate gauge redundancy and avoid redundant fit parameters, which further increases the implementation complexity.

These limitations motivate the development of a method that simultaneously satisfies:

- Manifest covariance in the definition, with no reliance on the COM.

- A complete and clean separation between orbital angular momentum and total spin.

- A general formula for partial wave amplitudes is applicable to particles of any spin, including the case with massless particles.

This is precisely the goal of the theoretical framework developed in this work. Its starting point is an on-shell formulation, where Lorentz covariance is manifest, and spin is encoded through $SU(2)$ little group covariance. In the on-shell framework, amplitudes are constructed directly in terms of spinor-helicity variables [56−62], which removes substantial gauge redundancy and trivializes the kinematical on-shell constraints, thereby making explicit calculations much more efficient. For massless particles, three-point amplitudes are entirely fixed by the consistency conditions from little group scaling [63, 64], while higher-point amplitudes can be determined from factorization, imposed by locality and unitarity, using gluing techniques or recursive relations [65−68].

The spinor-helicity formalism has subsequently been extended to processes involving massive particles [69−79]. In particular, Arkani-Hamed, Huang, and Huang [80] developed an on-shell, little group covariant representation of three-point amplitudes for particles of arbitrary mass and spin, formulated in terms of spinor-helicity variables. In this framework, the four-momentum of a massive particle is written as $p_{\alpha\dot{\alpha}} = \lambda_{\alpha I}\tilde{\lambda}^I_{\dot{\alpha}}$ with $SU(2)$ little group index I. With this decomposition, scattering amplitudes can be expressed entirely in terms of spinor variables. The resulting amplitudes are manifestly Lorentz invariant, transform covariantly under the little group, and can be calculated in any reference frame without additional boosts.

The present construction is inspired by this on-shell viewpoint and aims at building a covariant LS coupling, referred to as the covariant spinor method in canonical formalism (covariant canonical-spinor scheme). In this scheme, the angular momentum coupling is performed in the little group $SU(2)$ indices carried by the canonical-spinor variables. At the same time, the canonical-spinor variables carry SL(2,$\mathbb{C}$) Lorentz indices, ensuring covariance under general Lorentz transformations. As a result, the spin coupling does not require, as in the covariant tensor method, lifting the structure to the $SO(3,1)$ Lorentz representation and realizing it through contractions of Lorentz indices.

By contrast, in the traditional-LS method, both the spin coupling and the orbital angular momentum L are defined with respect to the little group associated with the parent standard momentum in the two-body COM frame. Therefore, in practical calculations, one needs to boost events back to the two-body COM frame. Here, a little group covariant map τ is constructed to map the little group representation of each daughter particle into the little group representation of the parent particle. The angular momenta are then coupled by the $SU(2)$ CGCs, which avoids an explicit boost back to the COM frame. Accordingly, the amplitude can be written as

$$\mathcal{A}(p_3,p_1,p_2;L,S) = \underbrace{\mathcal{Y}(p_3,p_1,p_2;L)}_{\text{orbital part}} \cdot \underbrace{\tau_1(p_3,p_1;s_3,s_1)\cdot\tau_2(p_3,p_2;s_3,s_2)}_{\text{spin part}}, \quad (5)$$

which can be evaluated directly from the four-momenta in any frame while preserving the traditional-LS partial wave term by term. Moreover, the scheme is formulated in a general formula applicable to particles of arbitrary spin, including the case with massless particles. Finally, this scheme simplifies practical PWA implementations for cascade decays with multiple decay chains. The amplitudes can be evaluated directly from the four-momenta in any frame while keeping an explicit LS separation.

This paper is organized as follows. In Section II, we review the definitions of canonical and helicity single-particle states and the corresponding constructions of two-particle states coupled to a total angular momentum J. We then construct the helicity and traditional-LS amplitudes, and compare their structures and mutual relations. In Section III, we systematically revisit the Zemach tensor method, the covariant tensor method, and the covariant projection tensor method in both the PS scheme and GS scheme, summarize their building blocks and implementation steps, and discuss their respective advantages and limitations in practical amplitude modeling. We also explain how these methods connect to the LS partial wave organization and how their results match the standard LS description in the two-body COM frame. In Section IV, we introduce the covariant spinor scheme in canonical formalism. The corresponding amplitudes are decomposed into LS partial waves and compared with the traditional-LS amplitudes in any frame, demonstrating their equivalence. We then reformulate the covariant projection tensor method in the PS scheme in terms of canonical-spinor variables and present several explicit examples of canonical-spinor amplitude calculations. In Section V, we apply covariant and non-covariant methods to generic cascade decays. In Section VI, we apply the traditional-LS amplitude, the helicity amplitude, and the canonical-spinor amplitude to the specific cascade decay $\Lambda_c^+ \to \Lambda\pi^+\pi^0$, and demonstrate their numerical consistency.

II. HELICITY AND CANONICAL STATES AND AMPLITUDES

A. Description of single-particle and two-particle states

1. *Single-particle state*

Relativistic single-particle states can be classified by unitary irreducible representations of the Poincaré group, known as the Wigner particle classification [81]. To describe the internal degrees of freedom on a given orbit, one first chooses a standard momentum k^μ and defines the corresponding little group as follows,

$$G_L = \{\Lambda \in SO^+(3,1) \mid \Lambda k = k\}. \quad (6)$$

For massive particles with $m>0$, one takes $k^\mu = (m,0,0,0)$. In this case, transformations that leave k invariant can only be spatial rotations, so the little group is $G_L \simeq SO(3)$. The spin is then labeled by an irreducible representation of $SO(3)$, namely spin-j with dimension $2j+1$. For massless particles with $m=0$, one takes

$k^\mu=(E,0,0,E)$. The little group that leaves this light-like momentum invariant is $SO(2)$. The internal degree of freedom is then labeled by an irreducible representation of $SO(2)$, which is one-dimensional and can be characterized by the helicity λ.

Next, we focus on the massive case, $m>0$, for which the little group is $SO(3)$. The spin degrees of freedom are therefore described by the usual spin-j irreducible representation of the rotation group. We choose a rest-frame single-particle state $|\mathbf{k},j\sigma\rangle$, where σ denotes the spin projection along the z-axis. The rotation generators J_i satisfy

$$[J_i,J_j]=\mathrm{i}\varepsilon_{ijk}J_k, \tag{7}$$

where i,j, and k run over x,y,z, and ε_{ijk} is the totally antisymmetric tensor. The state $|\mathbf{k},j\sigma\rangle$ satisfies the following relation with the angular momentum operator and can be written as

$$\begin{aligned} J^2|\mathbf{k},j\sigma\rangle &= j(j+1)|\mathbf{k},j\sigma\rangle,\\ J_z|\mathbf{k},j\sigma\rangle &= \sigma|\mathbf{k},j\sigma\rangle, \end{aligned} \tag{8}$$

where $J^2=J_x^2+J_y^2+J_z^2$. A finite rotation $R(\alpha,\beta,\gamma)$ on a physical system is associated with a transformation of the state $|\mathbf{k},j\sigma\rangle$. The corresponding unitary operator can be written as

$$U[R(\alpha,\beta,\gamma)]=\mathrm{e}^{-\mathrm{i}\alpha J_z}\mathrm{e}^{-\mathrm{i}\beta J_y}\mathrm{e}^{-\mathrm{i}\gamma J_z}, \tag{9}$$

where (α,β,γ) are the standard Euler angles. The unitary operator preserves the multiplication law,

$$U[R_2R_1]=U[R_2]U[R_1]. \tag{10}$$

When a unitary operator $U[R]$ acts on the state $|\mathbf{k},j\sigma\rangle$, it transforms as

$$U[R(\alpha,\beta,\gamma)]|\mathbf{k},j\sigma\rangle=\sum_{\sigma'}D^{\sigma'(j)}_{\sigma}(\alpha,\beta,\gamma)|\mathbf{k},j\sigma'\rangle, \tag{11}$$

where $D^{\sigma'(j)}_{\sigma}(\alpha,\beta,\gamma)$ is given by

$$\begin{aligned} D^{\sigma'(j)}_{\sigma}(\alpha,\beta,\gamma) &= \langle\mathbf{k},j\sigma'|U[R(\alpha,\beta,\gamma)]|\mathbf{k},j\sigma\rangle\\ &= \mathrm{e}^{-\mathrm{i}\sigma'\alpha}d^{\sigma'(j)}_{\sigma}(\beta)\mathrm{e}^{-\mathrm{i}\sigma\gamma}, \end{aligned} \tag{12}$$

and

$$d^{\sigma'(j)}_{\sigma}(\beta)=\langle\mathbf{k},j\sigma'|\mathrm{e}^{-\mathrm{i}\beta J_y}|\mathbf{k},j\sigma\rangle. \tag{13}$$

Our conventions and useful identities for $D^{\sigma'(j)}_{\sigma}$ and $d^{\sigma'(j)}_{\sigma}$ are given in Appendix A.

So far, we have specified the action of the rotation on rest states, which is governed by the generators J_i. To describe general Lorentz transformations, one needs the full set of infinitesimal generators of the homogeneous Lorentz group. In addition to spatial rotations, the Lorentz group contains boosts, generated by operators K_i. The six generators (J_i,K_i) satisfy the following commutation relations

$$[J_i,J_j]=\mathrm{i}\varepsilon_{ijk}J_k,\quad [J_i,K_j]=\mathrm{i}\varepsilon_{ijk}K_k,\quad [K_i,K_j]=-\mathrm{i}\varepsilon_{ijk}J_k. \tag{14}$$

A *pure* boost $B(\boldsymbol{\eta})$ is defined as the Lorentz transformation with no accompanying spatial rotation. For a *pure* boost of rapidity η along the direction $\hat{\mathbf{p}}\equiv\mathbf{p}/|\mathbf{p}|$, the corresponding unitary operator takes the form

$$U[B(\boldsymbol{\eta})]=\mathrm{e}^{-\mathrm{i}\eta\hat{\mathbf{p}}\cdot\mathbf{K}}, \tag{15}$$

where $E+|\mathbf{p}|=m\mathrm{e}^{\eta}$, $\eta=\mathrm{arctanh}\left(\frac{|\mathbf{p}|}{E}\right)$, and $\boldsymbol{\eta}=\hat{\mathbf{p}}\eta$. For a general Lorentz transformation, Λ may contain both rotations and boosts. The unitary operator of the general Lorentz transformation preserves the multiplication law,

$$U[\Lambda_2\Lambda_1]=U[\Lambda_2]\,U[\Lambda_1]. \tag{16}$$

With the Lorentz generators (J_i,K_i) specified, we can define relativistic single-particle states with momentum $p^\mu=(E,\mathbf{p})$ from the rest single-particle states. We introduce the **standard boost** $L(\mathbf{p})$ satisfying $p=L(\mathbf{p})k$, and define

$$|\mathbf{p},j\sigma\rangle\equiv U[L(\mathbf{p})]|\mathbf{k},j\sigma\rangle. \tag{17}$$

The choice of $L(\mathbf{p})$ is not unique; different choices correspond to different conventions for transporting the spin quantization axis from the rest frame to momentum $\mathbf{p}$. In the following, we focus on two widely used conventions: the canonical-standard boost and the helicity-standard boost.

The **canonical-standard boost** is taken to be a *pure* boost that sends k^μ to p^μ without introducing any rotation. One may write

$$L_c(\mathbf{p})\equiv B(\boldsymbol{\eta})=R(\phi,\theta,0)L_z(\eta)R^{-1}(\phi,\theta,0), \tag{18}$$

where (θ,ϕ) are the spherical angles of $\hat{\mathbf{p}}$, $L_z(\eta)$ is the boost along the z-axis, and the subscript c denotes the canonical-standard boost.

The **helicity-standard boost** is defined by first performing a rotation $R(\phi,\theta,0)$ and then performing a *pure* boost that sends k^μ to p^μ. Using the same rotation $R(\phi,\theta,0)$ that takes $\hat{z}$ into $\hat{\mathbf{p}}$, one defines

$$L_h(\mathbf{p}) \equiv B(\boldsymbol{\eta})R(\phi,\theta,0) = R(\phi,\theta,0)L_z(\eta), \tag{19}$$

where the subscript h denotes the helicity-standard boost. Although these two types of boosts yield the same standard momentum, the quantization axes of the resulting single-particle states are different. For the canonical standard boost, the quantization axis remains the fixed z-axis direction, while for the helicity-standard boost, the quantization axis aligns with the direction of the momentum. Their geometric representations are shown in Figs. 1 and 2.

The relation between the canonical-standard boost and the helicity-standard boost can be expressed as follows,

$$L_c(\mathbf{p}) = L_h(\mathbf{p})R^{-1}(\phi,\theta,0). \tag{20}$$

Based on the difference in the spin quantization axes, we define two types of single-particle states: the canonical particle state and the helicity particle state. For the canonical single-particle state, its quantization axis always points along the z-axis; therefore, it is obtained by applying the canonical standard boost to the single-particle state at rest. For the helicity single-particle state, its quantization axis always aligns with the direction of the momentum; hence, it is obtained by applying the helicity standard boost to the single-particle state at rest. The explicit expressions of these boosted states can be defined through Eqs. (18) and (19) as follows,

$$\begin{aligned}|\mathbf{p}, j\sigma\rangle &\equiv U[L_c(\mathbf{p})]|\mathbf{k}, j\sigma\rangle \\ &= U[R(\phi,\theta,0)]U[L_z(\eta)]U^{-1}[R(\phi,\theta,0)]|\mathbf{k}, j\sigma\rangle,\end{aligned} \tag{21}$$

and

$$\begin{aligned}|\mathbf{p}, j\lambda\rangle &\equiv U[L_h(\mathbf{p})]|\mathbf{k}, j\lambda\rangle \\ &= U[R(\phi,\theta,0)]U[L_z(\eta)]|\mathbf{k}, j\lambda\rangle.\end{aligned} \tag{22}$$

Due to the different definitions of the quantization axis for the two types of particle states, their transformation properties under rotations also differ. For the canonical single-particle state, because its spin quantization axis is always fixed along the z-direction, a rotation will mix its spin projection quantum numbers σ via a Wigner D-matrix. In contrast, for the helicity single-particle state, the quantization axis is always aligned with the momentum direction; after a rotation, the spin quantization axis simply follows the rotated momentum, so the spin projection along that axis remains unchanged. Their transformation rules under rotations are given by

$$\begin{aligned}U[R]|\mathbf{p}, j\sigma\rangle &= U[RR_0]U[L_z(\eta)]U^{-1}[RR_0]U[R]|\mathbf{k}, j\sigma\rangle \\ &= \sum_{\sigma'} D_{\sigma}^{\sigma'(j)}(R)|R\mathbf{p}, j\sigma'\rangle,\end{aligned} \tag{23}$$

and

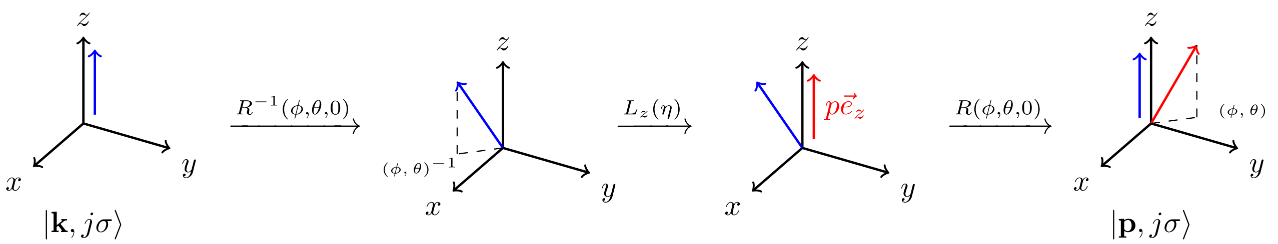

Fig. 1. (color online) Canonical-standard boost for a spin-j particle. The blue arrow indicates the direction of spin polarization, and the red arrow indicates the direction of momentum.

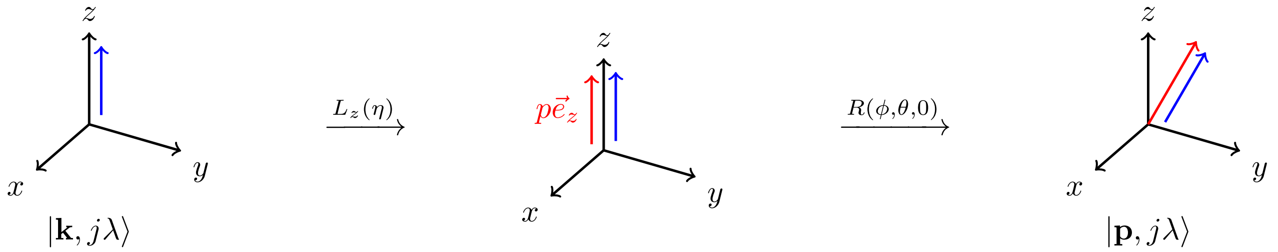

Fig. 2. (color online) Helicity-standard boost for a spin-j particle. The blue arrow indicates the direction of spin polarization, and the red arrow indicates the direction of momentum.

$$\begin{aligned}U[R]|\mathbf{p},j\lambda\rangle &= U[RR_0]U[L_z(\eta)]|\mathbf{k},j\lambda\rangle\\&= U[R'(\phi',\theta',\xi)]U[L_z(\eta)]|\mathbf{k},j\lambda\rangle\\&= U[R''(\phi',\theta',0)]U[L_z(\eta)]U[R_z(\xi(R,\mathbf{p}))]|\mathbf{k},j\lambda\rangle\\&= \mathrm{e}^{\mathrm{i}\lambda\xi(R,\mathbf{p})}|R\mathbf{p},j\lambda\rangle,\end{aligned}\tag{24}$$

where $R'(\phi',\theta',\xi)=RR_0$, and the azimuthal angle of the momentum $R\mathbf{p}$ is (ϕ',θ'). For the rotation $R'(\phi',\theta',\xi)$, the initial rotation about the z axis is not necessarily zero. Since this initial z-axis rotation does not change the final momentum direction, one can use the commutation relation in Eq. (14) to treat it as a rotation about the z axis applied on the state $|\mathbf{k},j\lambda\rangle$, which therefore produces the corresponding phase. It can be observed that helicity remains unchanged under rotations. The geometric difference between the canonical and helicity states is illustrated in Figs. 3 and 4.

The relation between the canonical state and the helicity state can be obtained from Eqs. (11), (19), (21), and (22) and is given by

$$\begin{aligned}|\mathbf{p},j\lambda\rangle &= U[R(\phi,\theta,0)]U[L_z(\eta)]U^{-1}\\&\quad\times[R(\phi,\theta,0)]U[R(\phi,\theta,0)]|\mathbf{k},j\lambda\rangle\\&= U[L_c(\mathbf{p})]U[R(\phi,\theta,0)]|\mathbf{k},j\lambda\rangle\\&= \sum_\sigma D^{\sigma(j)}_{\lambda}(R)|\mathbf{p},j\sigma\rangle.\end{aligned}\tag{25}$$

Having established the transformation properties of the relativistic canonical and helicity states under rotations, we now turn our attention to their behavior under Lorentz transformations. Considering an arbitrary four-momentum $p^\mu=(E,\mathbf{p})$, a proper homogeneous orthochronous Lorentz transformation $\Lambda^\mu_{\ \nu}$ transforms the four-momentum p^μ into p'^μ as follows,

$$p'^\mu=\Lambda^\mu_{\ \nu}p^\nu,\tag{26}$$

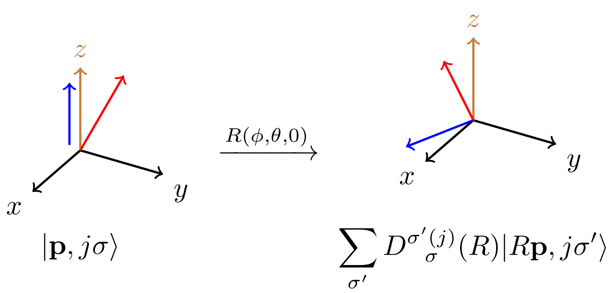

Fig. 3. (color online) Rotation acting on a canonical single-particle state. The brown line indicates the spin quantization axis, the red line represents the momentum direction of the particle, and the blue line denotes the spin polarization direction of the particle.

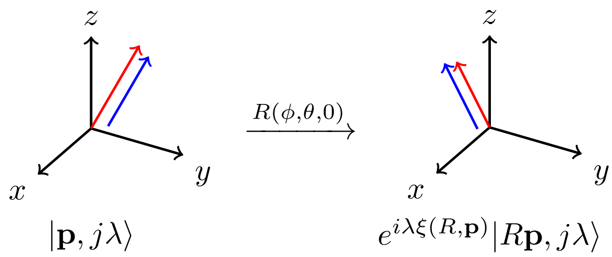

Fig. 4. (color online) Rotation acting on a helicity single-particle state. The red line indicates the momentum direction of the particle, which is also the spin quantization axis, and the blue line denotes the direction of the particle's spin polarization.

where the repeated indices are summed over. Under this Lorentz transformation, the state $U[\Lambda]|\mathbf{p},j\alpha\rangle$ is a linear combination of the states $|\mathbf{p}',j\alpha'\rangle$, namely

$$\begin{aligned}U[\Lambda]|\mathbf{p},j\alpha\rangle &= \sum_{\alpha'}C_{\alpha'\alpha}|\mathbf{p}',j\alpha'\rangle\\&= U[L(\mathbf{p}')]\sum_{\alpha'}C_{\alpha'\alpha}|\mathbf{k},j\alpha'\rangle,\end{aligned}\tag{27}$$

where α represents either the spin projection σ (canonical basis) or the helicity λ (helicity basis), with the boost $L(\mathbf{p})$ chosen accordingly. According to Eq. (11), the linear combination over α' represents a rotation acting on the single-particle state with standard momentum. This rotation is given by

$$\begin{aligned}U[\Lambda]|\mathbf{p},j\alpha\rangle &= U[\Lambda]U[L(\mathbf{p})]|\mathbf{k},j\alpha\rangle\\&= U[L(\mathbf{p}')]U\left[L^{-1}(\mathbf{p}')\Lambda L(\mathbf{p})\right]|\mathbf{k},j\alpha\rangle,\end{aligned}\tag{28}$$

with

$$W(\Lambda,p)\equiv L^{-1}(\mathbf{p}')\Lambda L(\mathbf{p}),\tag{29}$$

where $W(\Lambda,p)$ is a rotation that leaves the standard momentum k^μ invariant; namely, $W^\mu_{\ \nu}k^\nu=k^\mu$, and is referred to as the little group rotation. The action of this rotation on the state $|\mathbf{k},j\alpha\rangle$ is

$$U[W(\Lambda,p)]|\mathbf{k},j\alpha\rangle=\sum_{\alpha'}D^{\alpha'(j)}_{\alpha}(W(\Lambda,p))|\mathbf{k},j\alpha'\rangle,\tag{30}$$

where $D^{\alpha'(j)}_{\alpha}(W)$ is the little group representation matrix. Substituting this result into Eq. (28), the Lorentz transformation of the single-particle state becomes

$$U[\Lambda]|\mathbf{p},j\alpha\rangle=\sum_{\alpha'}D^{\alpha'(j)}_{\alpha}(W(\Lambda,p))|\mathbf{p}',j\alpha'\rangle.\tag{31}$$

This result illustrates how the canonical or helicity quantum numbers transform under Lorentz transformations. The little group rotation describes the additional Wigner rotation induced by the composition of non-collinear Lorentz boosts. When the Lorentz transformation is a pure boost $B(\boldsymbol{\eta}')$ whose direction is parallel to $\hat{\mathbf{p}}$, the transformed momentum remains collinear with p, which gives

$$\bar{p} = B(\boldsymbol{\eta}')p, \qquad \bar{p} \parallel p. \tag{32}$$

Using Eqs. (18) and (19), the canonical little group rotation is

$$\begin{aligned} W_c(B(\boldsymbol{\eta}'), p) &= L_c^{-1}(\bar{\mathbf{p}})B(\boldsymbol{\eta}')L_c(\mathbf{p}) \\ &= R(\phi,\theta,0)L_z(-\eta'-\eta)L_z(\eta')L_z(\eta)R^{-1}(\phi,\theta,0) \\ &= \mathbf{I}, \end{aligned} \tag{33}$$

and the helicity little group rotation is

$$\begin{aligned} W_h(B(\boldsymbol{\eta}'), p) &= L_h^{-1}(\bar{\mathbf{p}})B(\boldsymbol{\eta}')L_h(\mathbf{p}) \\ &= L_z(-\eta'-\eta)L_z(\eta')L_z(\eta) \\ &= \mathbf{I}. \end{aligned} \tag{34}$$

Therefore, when the Lorentz transformation is a pure boost along the momentum direction $\hat{\mathbf{p}}$, the canonical and helicity little group rotations are both the identity $\mathbf{I}$. That is, for the corresponding single-particle states under the Lorentz transformation, the canonical spin projection σ and the helicity λ remain unchanged, which can be written as

$$U[B(\boldsymbol{\eta}')]|\mathbf{p}, j\alpha\rangle = |\bar{\mathbf{p}}, j\alpha\rangle. \tag{35}$$

When the Lorentz transformation is a pure rotation R, the canonical little group rotation is

$$\begin{aligned} W_c(R, p) &= L_c^{-1}(R\mathbf{p})RL_c(\mathbf{p}) \\ &= L_c^{-1}(R\mathbf{p})L_c(R\mathbf{p})R \\ &= R, \end{aligned} \tag{36}$$

where for canonical states under a pure rotation, the corresponding little group rotation is R. Therefore, the spin projection σ does not remain unchanged, with the action of the rotation given in Eq. (23). In contrast, the helicity little group rotation is

$$\begin{aligned} W_h(R, p) &= L_h^{-1}(R\mathbf{p})RL_h(\mathbf{p}) = L_h^{-1}(R\mathbf{p})RR_0L_z(\eta) \\ &= L_h^{-1}(R\mathbf{p})R'(\phi',\theta',\xi)L_z(\eta) \\ &= L_h^{-1}(R\mathbf{p})\Big(L_h(R\mathbf{p})R_z(\xi(R,\mathbf{p}))\Big) \\ &= R_z(\xi(R,\mathbf{p})), \end{aligned} \tag{37}$$

where $R_z(\xi(R,\mathbf{p}))$ is a rotation about the helicity direction, which produces a phase and does not change the helicity, with the corresponding result given in Eq. (24). Finally, for both the canonical and helicity single particle states, the pure boost along the momentum direction leaves the spin projection σ and helicity λ unchanged. Under pure rotation, the helicity λ is preserved and the state transforms only by a phase, whereas the canonical spin projection σ generally mixes.

2. *Construction of two-particle states*

Having introduced the canonical and helicity single-particle states, we now consider a two-body system of two massive particles, particle-1 and particle-2, with spins s_1 and s_2, and construct total angular momentum states with angular momentum J in either the traditional-LS basis or the helicity basis.

In the traditional-LS construction, a common spin quantization axis is fixed, conventionally the z-axis, and the relative orbital angular momentum $\boldsymbol{L}$ is defined with respect to the same axis. The total spin is obtained by adding the individual spins, $\boldsymbol{S} = \boldsymbol{s}_1 + \boldsymbol{s}_2, S_z = s_{1z} + s_{2z}$. There are $(2s_1+1)\times(2s_2+1)$ subspaces that are described entirely by the angular. Here, the angular degrees of freedom and the $(2s_1+1)\times(2s_2+1)$ spin degrees of freedom are independent. Consequently, the states obtained from the D-matrix, namely spherical harmonic, decomposition exhibit a $(2s_1+1)\times(2s_2+1)$-fold degeneracy, which must be resolved by further decomposition using CGCs. Thus, the total angular momentum J is then built through two successive CGC couplings: first coupling $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ to form $\boldsymbol{S}$, and then coupling $\boldsymbol{L}$ with $\boldsymbol{S}$ to obtain $\boldsymbol{J}$. This construction is based on the complete set of commuting observables $\{\boldsymbol{J}^2, J_z, \boldsymbol{L}^2, \boldsymbol{S}^2, P^2, \boldsymbol{P}\}$. In the subsequent discussion, we focus only on the construction of two-particle states in the COM frame; therefore, we omit $P^2, \boldsymbol{P}$, and denote the common eigenstates as $|JMLS\rangle$. By inserting the completeness relation of the product basis formed by the two canonical single-particle spin states, $|JMLS\rangle$ can be written explicitly as

$$\begin{aligned} |JMLS\rangle &= \sum_{\sigma_1\sigma_2}|\Omega\sigma_1\sigma_2\rangle\langle\Omega\sigma_1\sigma_2|JMLS\rangle \\ &= \sum_{\sigma_1\sigma_2\sigma_L\sigma_S} C_{J,M}^{L,\sigma_L;S,\sigma_S} C_{S,\sigma_S}^{s_1,\sigma_1;s_2,\sigma_2} \int \mathrm{d}\Omega\, Y_{\sigma_L}^L(\Omega)|\Omega\sigma_1\sigma_2\rangle, \end{aligned} \tag{38}$$

where $C^{L,\sigma_L;S,\sigma_S}_{J,M}$ and $C^{s_1,\sigma_1;s_2,\sigma_2}_{S,\sigma_S}$ are the CGCs.

By contrast, in the helicity construction, the spin quantization axis of each particle is chosen to be along its own momentum direction. When the particle state is rotated, its quantization axis remains aligned with the rotated momentum direction; therefore, helicity possesses rotational invariance. In this construction, the complete set of commuting observables is $\{\boldsymbol{J}^2, J_z, \lambda_1, \lambda_2, P^2, \boldsymbol{P}\}$, and the common eigenstates are denoted as $|JM\lambda_1\lambda_2\rangle$. When the helicity is fixed, the rotational properties of the entire particle system in the COM frame are completely determined by the momentum direction of particle-1 and its helicity projection along that direction, $\lambda = \lambda_1 - \lambda_2$. In the subspace of fixed λ_i, only one rotational degree of freedom remains to characterize the state. Therefore, we use the completeness of the D-matrices to directly decompose the state into components of different angular momenta as

$$|JM\lambda_1\lambda_2\rangle = \sum_{\lambda'_1\lambda'_2} |\phi\theta\lambda'_1\lambda'_2\rangle\langle\phi\theta\lambda'_1\lambda'_2|JM\lambda_1\lambda_2\rangle = N_J \int \mathrm{d}\Omega D^{M(J)*}_{\lambda}(\phi,\theta,0)|\phi\theta\lambda_1\lambda_2\rangle. \tag{39}$$

In this way, the helicity construction does not require an explicit LS coupling. Next, we will present the explicit construction procedure.

Canonical construction The canonical construction is formulated in terms of the canonical two-particle state in the COM frame of particles 1 and 2. Consider two massive particles, with particle-1 and particle-2 having spins s_1 and s_2, respectively. The canonical two-particle state is defined by

$$|\phi\theta\sigma_1\sigma_2\rangle \equiv NU[L_{1,c}(\mathbf{p})]|\mathbf{k}_1, s_1\sigma_1\rangle U[L_{2,c}(-\mathbf{p})]|\mathbf{k}_2, s_2\sigma_2\rangle, \tag{40}$$

where N is the normalization constant, $U[L_{i,c}(\mathbf{p})]$ $(i=1,2)$ is the canonical-standard boost in Eq. (18), and the numerical subscripts are used to distinguish the canonical-standard boosts for different particles. The explicit calculation of the normalization constant N is given in Appendix B. The two-particle state $|\phi\theta\sigma_1\sigma_2\rangle$ satisfies the normalization condition, which can be written as

$$\langle\phi'\theta'\sigma'_1\sigma'_2|\phi\theta\sigma_1\sigma_2\rangle = \delta(\phi'-\phi)\delta(\theta'-\theta)\delta^{\sigma'_1}_{\sigma_1}\delta^{\sigma'_2}_{\sigma_2}. \tag{41}$$

By coupling the spins s_1 and s_2 in Eq. (40), we can define two-particle states with total spin S as

$$|\phi\theta S\sigma_S\rangle = \sum_{\sigma_1\sigma_2} C^{s_1,\sigma_1;s_2,\sigma_2}_{S,\sigma_S}|\phi\theta\sigma_1\sigma_2\rangle, \tag{42}$$

where $C^{s_1,\sigma_1;s_2,\sigma_2}_{S,\sigma_S}$ is the CGC. Using the angular momentum selection rules, one obtains

$$|s_1 - s_2| \le S \le s_1 + s_2. \tag{43}$$

Describing the total angular momentum J of the two-particle system requires consideration of both the spins of the particles and their relative orbital motion. In the COM frame, the direction of relative momentum $\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2$ between the two particles determines the angular dependence of the system, which can be expanded in spherical harmonics to describe orbital angular momentum states.

Starting from the state $|\phi\theta S\sigma_S\rangle$, where the direction of the relative momentum is specified by the angles (θ,ϕ), the orbital angular momentum eigenstates are defined by projecting $|\phi\theta S\sigma_S\rangle$ onto the normalized spherical harmonic $Y^L_{\sigma_L}(\theta,\phi)$. Explicitly, this is defined as

$$|L\sigma_L S\sigma_S\rangle \equiv \int \mathrm{d}\Omega Y^L_{\sigma_L}(\Omega)|\Omega S\sigma_S\rangle, \tag{44}$$

where $\Omega = (\theta,\phi)$ and $\mathrm{d}\Omega = \mathrm{d}\phi\,\mathrm{d}\cos\theta$, and the spherical harmonic $Y^L_{\sigma_L}(\theta,\phi)$ is defined in Eq. (A13).

The state $|L\sigma_L S\sigma_S\rangle$ now carries both the orbital angular momentum L and the total spin S. Under spatial rotations, it transforms according to the direct product representation of the rotation group: the orbital part transforms under $D^L(R)$, and the spin part under $D^S(R)$. This can be expressed as

$$U[R]|L\sigma_L S\sigma_S\rangle = \sum_{\sigma'_L\sigma'_S} D^{\sigma'_L(L)}_{\sigma_L}(R) D^{\sigma'_S(S)}_{\sigma_S}(R)|L\sigma'_L S\sigma'_S\rangle. \tag{45}$$

The orbital angular momentum L and the total spin S can now be coupled to obtain the total angular momentum J, which can be written as

$$|JMLS\rangle = \sum_{\sigma_L\sigma_S} C^{L,\sigma_L;S,\sigma_S}_{J,M}|L\sigma_L S\sigma_S\rangle = \sum_{\sigma_1\sigma_2\sigma_L\sigma_S} C^{L,\sigma_L;S,\sigma_S}_{J,M} C^{s_1,\sigma_1;s_2,\sigma_2}_{S,\sigma_S} \int \mathrm{d}\Omega\, Y^L_{\sigma_L}(\Omega)|\Omega\sigma_1\sigma_2\rangle. \tag{46}$$

Using the angular momentum selection rules, the following constraints are obtained:

$$|L-S| \le J \le L+S, \quad \sigma_S = \sigma_1 + \sigma_2, \quad \sigma_L = M - \sigma_S. \tag{47}$$

From the normalization condition in Eq. (41), the states $|JMLS\rangle$ satisfy

$$\langle J'M'L'S'|JMLS\rangle = \delta_J^{J'}\delta_M^{M'}\delta_L^{L'}\delta_S^{S'}, \tag{48}$$

and the completeness relation is given by

$$\sum_{JMLS}|JMLS\rangle\langle JMLS| = \mathbf{I}. \tag{49}$$

The transformation properties of the state $|JMLS\rangle$ under rotations are identical to those of a single-particle state at rest. Its transformation can be expressed as follows,

$$U[R]|JMLS\rangle = \sum_{M'} D_M^{M'(J)}(R)|JM'LS\rangle, \tag{50}$$

where $D_M^{M'(J)}(R)$ is the Wigner D-matrix for angular momentum J. This transformation property shows that $|JMLS\rangle$ forms a complete set of eigenstates of the angular momentum operators $\mathbf{J}^2$ and J_z, with the transformation behavior completely determined by the total angular momentum J.

Helicity construction The helicity construction is formulated in terms of the helicity two-particle state in the COM frame. For two massive particles with helicities λ_1 and λ_2, the helicity two-particle state can be defined as

$$\begin{aligned}|\phi\theta\lambda_1\lambda_2\rangle &\equiv NU[L_{1,h}(\mathbf{p})]|\mathbf{k}_1,s_1\lambda_1\rangle U[L_{2,h}(-\mathbf{p})]|\mathbf{k}_2,s_2\ \lambda_2\rangle\\ &= NU[R]\Big(U[L_z(\eta_1)]|\mathbf{k}_1,s_1\lambda_1\rangle U[L_z(-\eta_2)]|\mathbf{k}_2,s_2\ \lambda_2\rangle\Big)\\ &= NU[R]\Big(|\mathbf{p}_z,s_1\lambda_1\rangle|-\mathbf{p}_z,s_2\ \lambda_2\rangle\Big)\\ &= NU[R]|00\lambda_1\lambda_2\rangle,\end{aligned} \tag{51}$$

where N is the normalization constant, $U[L_{i,c}(\mathbf{p})]$ $(i=1,2)$ is the helicity-standard boost in Eq. (19), and the numerical subscripts are used to distinguish the helicity-standard boosts for different particles. Due to the rotational invariance of the helicities λ_1 and λ_2, the action of a rotation R' on Eq. (51) gives

$$\begin{aligned}U[R']|\Omega\lambda_1\lambda_2\rangle &= NU[R'R]|00\lambda_1\lambda_2\rangle\\ &= |R'\Omega\lambda_1\lambda_2\rangle\\ &= |R''\lambda_1\lambda_2\rangle,\end{aligned} \tag{52}$$

where $\Omega=(\theta,\phi)$ and $R''=R'\Omega$.

When the helicities (λ_1,λ_2) are fixed, the nontrivial rotational dependence of the two-particle state in the COM frame is entirely encoded in the momentum direction of particle-1, parametrized by $\Omega=(\phi,\theta)$, together with the helicity difference $\lambda=\lambda_1-\lambda_2$. A rotation about the momentum axis leaves the helicities invariant and only produces a phase $e^{-i\lambda\alpha}$; hence the third Euler angle can be chosen as 0. Therefore, using the completeness of the Wigner D-functions for fixed right index λ, we project the angle state $|\phi\theta\lambda_1\lambda_2\rangle$ onto definite total angular momentum J and its z-component M:

$$|JM\lambda_1\lambda_2\rangle = N_J\int \mathrm{d}\Omega\, D_\lambda^{M(J)*}(\phi,\theta,0)\,|\phi\theta\lambda_1\lambda_2\rangle. \tag{53}$$

Using Eqs. (52) and (53), the transformation of the state $|JM\lambda_1\lambda_2\rangle$ under rotations is given by

$$\begin{aligned}U[R']|JM\lambda_1\lambda_2\rangle &= N_J\int \mathrm{d}\Omega D_\lambda^{M(J)*}(\phi,\theta,0)U[R']|\phi\theta\lambda_1\lambda_2\rangle\\ &= N_J\int \mathrm{d}\Omega D_\lambda^{M(J)*}(\Omega)|R''\lambda_1\lambda_2\rangle,\end{aligned} \tag{54}$$

where the Wigner D-matrix $D_\lambda^{*M(J)}(\Omega)$ can be written as

$$\begin{aligned}D_\lambda^{M(J)*}(\Omega) &= D_\lambda^{M(J)*}(R'^{-1}R'')\\ &= \sum_{M'} D_{M'}^{M(J)*}(R'^{-1})D_\lambda^{M'(J)*}(R'')\\ &= \sum_{M'} D_M^{M'(J)}(R')D_\lambda^{M'(J)*}(R'').\end{aligned} \tag{55}$$

Using this relation, one obtains

$$U[R']|JM\lambda_1\lambda_2\rangle = \sum_{M'} D_M^{M'(J)}(R')|JM'\lambda_1\lambda_2\rangle. \tag{56}$$

Thus, the state $|JM\lambda_1\lambda_2\rangle$ with definite total angular momentum J is constructed by coupling the two particles through their helicities λ_1 and λ_2. The state $|JM\lambda_1\lambda_2\rangle$ satisfies the normalization condition,

$$\langle J'M'\lambda_1'\lambda_2'|JM\lambda_1\lambda_2\rangle = \delta_J^{J'}\delta_M^{M'}\delta_{\lambda_1}^{\lambda_1'}\delta_{\lambda_2}^{\lambda_2'}, \tag{57}$$

which fixes the normalization constant to be

$$N_J = \sqrt{\frac{2J+1}{4\pi}}. \tag{58}$$

Finally, the completeness relation takes the form:

$$\sum_{JM\lambda_1\lambda_2}|JM\lambda_1\lambda_2\rangle\langle JM\lambda_1\lambda_2| = \mathbf{I}. \tag{59}$$

Equivalence between two constructions Having introduced both the helicity and canonical constructions, we now demonstrate their equivalence. Despite their different starting points, both formulations yield states with total angular momentum J. In the following, we derive the explicit transformation between these two equivalent constructions. Starting from the respective two-particle states and using Eqs. (19), (25), and (40), we have

$$\begin{aligned}
&|\phi\theta\lambda_1\lambda_2\rangle\\
&= NU[L_{1,h}(\mathbf{p})]|\mathbf{k}_1, s_1\lambda_1\rangle U[L_{2,h}(-\mathbf{p})]|\mathbf{k}_2, s_2 -\lambda_2\rangle\\
&= NU[L_{1,c}(\mathbf{p})]U[R]|\mathbf{k}_1, s_1\lambda_1\rangle U[L_{2,c}(-\mathbf{p})]U[R]|\mathbf{k}_2, s_2 -\lambda_2\rangle\\
&= \sum_{\sigma_1\sigma_2} D^{\sigma_1(s_1)}_{\lambda_1}(\phi,\theta,0) D^{\sigma_2(s_2)}_{-\lambda_2}(\phi,\theta,0) U\\
&\quad\times [L_{1,c}(\mathbf{p})]|\mathbf{k}_1, s_1\sigma_1\rangle U[L_{2,c}(-\mathbf{p})]|\mathbf{k}_2, s_2\sigma_2\rangle\\
&= \sum_{\sigma_1\sigma_2} D^{\sigma_1(s_1)}_{\lambda_1}(\phi,\theta,0) D^{\sigma_2(s_2)}_{-\lambda_2}(\phi,\theta,0)|\phi\theta\sigma_1\sigma_2\rangle.
\end{aligned} \tag{60}$$

Substituting this into the definition of the total angular momentum state in Eq. (53) yields

$$\begin{aligned}
|JM\lambda_1\lambda_2\rangle &= N_J\int \mathrm{d}\Omega D^{M(J)*}_{\lambda}(\phi,\theta,0)|\phi\theta\lambda_1\lambda_2\rangle\\
&= N_J\sum_{\sigma_1\sigma_2}\int \mathrm{d}\Omega D^{M(J)*}_{\lambda}(\phi,\theta,0) D^{\sigma_1(s_1)}_{\lambda_1}(\phi,\theta,0)\\
&\quad\times D^{\sigma_2(s_2)}_{-\lambda_2}(\phi,\theta,0)|\phi\theta\sigma_1\sigma_2\rangle.
\end{aligned} \tag{61}$$

Using Eqs. (A14), (A16), and (A13), coupling the three Wigner D-matrices via CGCs yields

$$D^{\sigma_1(s_1)}_{\lambda_1} D^{\sigma_2(s_2)}_{-\lambda_2} = \sum_S C^{s_1,\sigma_1;s_2,\sigma_2}_{S,\sigma_S} C^{S,\lambda}_{s_1,\lambda_1;s_2,-\lambda_2} D^{\sigma_S(S)}_{\lambda}, \tag{62}$$

and

$$\begin{aligned}
D^{M(J)*}_{\lambda} D^{\sigma_S(S)}_{\lambda} &= \sum_L \left(\frac{2L+1}{2J+1}\right) C^{L,\sigma_L;S,\sigma_S}_{J,M} C^{J,\lambda}_{L,0;S,\lambda} D^{\sigma_L(L)*}_{0}\\
&= \sum_L \sqrt{\frac{4\pi}{2L+1}}\left(\frac{2L+1}{2J+1}\right) C^{L,\sigma_L;S,\sigma_S}_{J,M} C^{J,\lambda}_{L,0;S,\lambda} Y^L_{\sigma_L},
\end{aligned} \tag{63}$$

where

$$|s_1 - s_2| \le S \le s_1 + s_2, \quad |J-S| \le L \le J+S, \tag{64}$$

$$\sigma_S = \sigma_1 + \sigma_2, \quad \lambda = \lambda_1 - \lambda_2, \quad M = \sigma_L + \sigma_S, \tag{65}$$

and $Y^L_{\sigma_L}$ is the normalized spherical harmonic. Combining Eqs. (61), (62), and (63), we obtain the transformation between helicity and canonical constructions as

$$|JM\lambda_1\lambda_2\rangle = \sum_{LS}\left(\frac{2L+1}{2J+1}\right)^{\frac{1}{2}} C^{J,\lambda}_{L,0;S,\lambda} C^{S,\lambda}_{s_1,\lambda_1;s_2,-\lambda_2}|JMLS\rangle, \tag{66}$$

and the recoupling coefficient

$$\langle J'M'LS|JM\lambda_1\lambda_2\rangle = \left(\frac{2L+1}{2J+1}\right)^{\frac{1}{2}} C^{J,\lambda}_{L,0;S,\lambda} C^{S,\lambda}_{s_1,\lambda_1;s_2,-\lambda_2}\delta^{J'}_{J}\delta^{M'}_{M}. \tag{67}$$

The inverse relation of Eq. (66) is

$$|JMLS\rangle = \sum_{\lambda_1\lambda_2}\left(\frac{2L+1}{2J+1}\right)^{\frac{1}{2}} C^{L,0;S,\lambda}_{J,\lambda} C^{s_1,\lambda_1;s_2,-\lambda_2}_{S,\lambda}|JM\lambda_1\lambda_2\rangle. \tag{68}$$

B. Basic amplitudes for two-body decay

In the following discussion, we consider the two-body decay process $3\to 1+2$, where all particles are massive, and particle-1, particle-2, and particle-3 carry spins s_1, s_2, and s_3, respectively. We then compare the canonical and helicity representations in the COM frame. Finally, we present the two-body decay amplitude in any frame.

1. *Canonical and helicity decay amplitudes and their equivalence*

The canonical decay amplitude in the COM frame, denoted by $A^{\sigma_1\sigma_2}_{\sigma_3}$, can be expressed as

$$A^{\sigma_1\sigma_2}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}^*_1,\mathbf{p}^*_2) = \langle \mathbf{p}^*_1, s_1\sigma_1;\mathbf{p}^*_2, s_2\sigma_2|M|\mathbf{k}_3, s_3\sigma_3\rangle, \quad \mathbf{p}^*_1 = -\mathbf{p}^*_2. \tag{69}$$

Within the LS coupling formalism, this amplitude can be expanded in LS partial waves. Using Eq. (46), one has

$$\begin{aligned}
&A^{\sigma_1\sigma_2}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}^*_1,\mathbf{p}^*_2)\\
&= \sum_{LS}\frac{1}{N}\langle\Omega\sigma_1\sigma_2|s_3\sigma_3LS\rangle\langle s_3\sigma_3LS|M|s_3\sigma_3\rangle\\
&= \sum_{LS} C^{L,\sigma_L;S,\sigma_S}_{s_3,\sigma_3} C^{s_1,\sigma_1;s_2,\sigma_2}_{S,\sigma_S} Y^L_{\sigma_L}(\Omega)\frac{1}{N}\langle s_3\sigma_3LS|M|s_3\sigma_3\rangle\\
&= \sum_{LS} g_{LS} C^{L,\sigma_L;S,\sigma_S}_{s_3,\sigma_3} C^{s_1,\sigma_1;s_2,\sigma_2}_{S,\sigma_S} Y^L_{\sigma_L}(\Omega),
\end{aligned} \tag{70}$$

where g_{LS} is called the LS coupling amplitude and is determined from experiments. It can be written as

$$g_{LS} = \frac{1}{N}\langle s_3\sigma_3LS|M|s_3\sigma_3\rangle. \tag{71}$$

From the angular momentum selection rules, the following constraints hold:

$$|s_1 - s_2| \le S \le s_1 + s_2, \quad |s_3 - S| \le L \le s_3 + S. \tag{72}$$

The traditional-LS decay amplitude is defined as

$$A^{\sigma_1\sigma_2}_{\sigma_3}(\mathbf{k}_3, \mathbf{p}^*_1, \mathbf{p}^*_2; L, S) \equiv g_{LS} C^{L,\sigma_L;S,\sigma_S}_{s_3,\sigma_3} C^{s_1,\sigma_1;s_2,\sigma_2}_{S,\sigma_S} Y^L_{\sigma_L}(\Omega). \tag{73}$$

Turning to the helicity coupling, the helicity decay amplitude in the COM frame is defined as

$$A^{\lambda_1\lambda_2}_{\sigma_3}(\mathbf{k}_3, \mathbf{p}^*_1, \mathbf{p}^*_2) = \langle \mathbf{p}^*_1, s_1\lambda_1; \mathbf{p}^*_2, s_2\lambda_2 | M | \mathbf{k}_3, s_3\sigma_3 \rangle. \tag{74}$$

In this definition, the quantization axis for the initial state (particle-3) is chosen along its spin direction. Consequently, in its rest frame, the helicity λ_3 is identical to the spin projection σ_3, which is why σ_3 is used in the amplitude. Using Eqs. (53) and (66), the amplitude can be written as

$$\begin{aligned} A^{\lambda_1\lambda_2}_{\sigma_3} &= \frac{1}{N} \langle \Omega\lambda_1\lambda_2 | s_3\sigma_3\lambda_1\lambda_2 \rangle \langle s_3\sigma_3\lambda_1\lambda_2 | M | s_3\sigma_3 \rangle \\ &= \sqrt{\frac{2s_3+1}{4\pi}} D^{\sigma_3(s_3)*}_{\lambda}(\phi, \theta, 0) H^{\lambda_1\lambda_2}, \end{aligned} \tag{75}$$

where θ and ϕ are the helicity angles of particle-1 in the rest frame of particle-3, and $H^{\lambda_1\lambda_2}$ is called the helicity coupling amplitude. It can be expressed as

$$H^{\lambda_1\lambda_2} = \frac{1}{N} \langle s_3\sigma_3\lambda_1\lambda_2 | M | s_3\sigma_3 \rangle. \tag{76}$$

Both the canonical decay amplitude and the helicity decay amplitude can be expressed in the LS coupling formalism. The canonical decay amplitude discussed above has already been written in terms of its LS components. For the helicity decay amplitude, using Eqs. (49) and (66), the helicity coupling amplitude in the LS coupling formalism [29] can be expressed as follows,

$$\begin{aligned} H^{\lambda_1\lambda_2} &= \frac{1}{N} \langle s_3\sigma_3\lambda_1\lambda_2 | M | s_3\sigma_3 \rangle \\ &= \sum_{LS} \langle s_3\sigma_3\lambda_1\lambda_2 | s_3\sigma_3 LS \rangle \frac{1}{N} \langle s_3\sigma_3 LS | M | s_3\sigma_3 \rangle \\ &= \sum_{LS} g_{LS} \left(\frac{2L+1}{2s_3+1} \right)^{\frac{1}{2}} C^{L,0;S,\lambda}_{J,\lambda} C^{s_1,\lambda_1;s_2,-\lambda_2}_{S,\lambda}. \end{aligned} \tag{77}$$

From the normalization relation, it follows that

$$\sum_{\lambda_1\lambda_2} |H^{\lambda_1\lambda_2}|^2 = \sum_{LS} |g_{LS}|^2. \tag{78}$$

Accordingly, the helicity coupling amplitude for a specific LS partial wave is defined as

$$H^{\lambda_1\lambda_2}(L, S) \equiv g_{LS} \left(\frac{2L+1}{2s_3+1} \right)^{\frac{1}{2}} C^{L,0;S,\lambda}_{J,\lambda} C^{s_1,\lambda_1;s_2,-\lambda_2}_{S,\lambda}. \tag{79}$$

The difference between the helicity decay amplitude and the canonical decay amplitude is that they are expressed in different bases; nevertheless, they are physically equivalent. By redefining the canonical state using Eqs. (25) and (60), the helicity amplitude can be written as

$$\begin{aligned} A^{\lambda_1\lambda_2}_{\sigma_3} &= \langle \mathbf{p}^*_1, s_1\lambda_1; \mathbf{p}^*_2, s_2\lambda_2 | M | \mathbf{k}_3, s_3\sigma_3 \rangle \\ &= \sum_{\sigma_1\sigma_2} D^{\sigma_1(s_1)*}_{\lambda_1}(\phi, \theta, 0) D^{\sigma_2(s_2)*}_{-\lambda_2}(\phi, \theta, 0) \langle \\ &\quad \times \mathbf{p}^*_1, s_1\sigma_1; \mathbf{p}^*_2, s_2\sigma_2 | M | \mathbf{k}_3, s_3\sigma_3 \rangle \\ &= \sum_{\sigma_1\sigma_2} D^{\sigma_1(s_1)*}_{\lambda_1}(\phi, \theta, 0) D^{\sigma_2(s_2)*}_{-\lambda_2}(\phi, \theta, 0) A^{\sigma_1\sigma_2}_{\sigma_3} \\ &= \sum_{\sigma_1\sigma_2} (-1)^{s_2} D^{\sigma_1(s_1)*}_{\lambda_1}(\phi, \theta, 0) D^{\sigma_2(s_2)*}_{\lambda_2} \\ &\quad \times (\pi+\phi, \pi-\theta, 0) A^{\sigma_1\sigma_2}_{\sigma_3}. \end{aligned} \tag{80}$$

2. *Example of helicity amplitude calculation*

In this subsubsection, we provide some calculation examples of helicity amplitudes. For the example of traditional-LS amplitude, to facilitate a comparison with other LS methods, it is presented in subsection III.C.

In a fixed three-point amplitude, the helicity-coupling amplitude depends only on the masses. Therefore, for a given two-body decay process with definite masses and spins, it is merely a constant factor. Hence, we are only concerned with the angular dependence. Finally, we will also illustrate what the helicity-coupling amplitude should be for specific interaction vertices.

The energy and the momentum of the daughter particles are given as

$$E_1 = \frac{m_3^2 + m_1^2 - m_2^2}{2m_3}, \quad E_2 = \frac{m_3^2 + m_2^2 - m_1^2}{2m_3}, \tag{81}$$

$$q = |\mathbf{p}^*_1| = |\mathbf{p}^*_2| = \frac{\sqrt{((m_3+m_2)^2 - m_1^2)((m_3+m_1)^2 - m_2^2)}}{2m_3}, \tag{82}$$

where m_1, m_2, and m_3 are the masses of the particles.

For the two-body decay process $3 \to 1+2$ with spins $s_3 = 1$, $s_1 = 1$, $s_2 = 0$, the helicity amplitude is

$$A_{\lambda_3}^{\lambda_1\lambda_2} = H^{\lambda_1\lambda_2} D_{\lambda}^{\lambda_3(1)*}(\phi,\theta,0), \tag{83}$$

where $\lambda = \lambda_1 - \lambda_2$. When the decay vertex is $C_{VVS}\, g_{\mu\nu} V_3^{\mu} V_1^{\nu} S$, where C_{VVS} is a constant, the above amplitude can be written as

$$\begin{aligned}
A_{+}^{+0} &= [-C_{VVS}]\, D_{+}^{+(1)*}(\phi,\theta,0),\\
A_{+}^{00} &= \left[-C_{VVS}\frac{E_1}{m_1}\right] D_{0}^{+(1)*}(\phi,\theta,0),\\
A_{+}^{-0} &= [-C_{VVS}]\, D_{-}^{+(1)*}(\phi,\theta,0),\\
A_{0}^{+0} &= [-C_{VVS}]\, D_{+}^{0(1)*}(\phi,\theta,0),\\
A_{0}^{00} &= \left[-C_{VVS}\frac{E_1}{m_1}\right] D_{0}^{0(1)*}(\phi,\theta,0),\\
A_{0}^{-0} &= [-C_{VVS}]\, D_{-}^{0(1)*}(\phi,\theta,0),\\
A_{-}^{+0} &= [-C_{VVS}]\, D_{+}^{-(1)*}(\phi,\theta,0),\\
A_{-}^{00} &= \left[-C_{VVS}\frac{E_1}{m_1}\right] D_{0}^{-(1)*}(\phi,\theta,0),\\
A_{-}^{-0} &= [-C_{VVS}]\, D_{-}^{-(1)*}(\phi,\theta,0).
\end{aligned} \tag{84}$$

By comparing Eqs. (83) and (84), the helicity-coupling amplitude can be determined.

For the two-body decay process $3 \to 1+2$ with spins $s_3 = \frac{1}{2}, s_1 = \frac{1}{2}, s_2 = 0$, the helicity amplitude is

$$A_{\lambda_3}^{\lambda_1\lambda_2} = H^{\lambda_1\lambda_2} D_{\lambda}^{\lambda_3(\frac{1}{2})*}(\phi,\theta,0), \tag{85}$$

where $\lambda = \lambda_1 - \lambda_2$. When the decay vertex is $\bar{f}_1 \gamma^{\mu}(C_L P_L + C_R P_R) f_2 S$, where C_L and C_R are constants. The left and right-handed projection operators are $P_{L,R}$. The above amplitude can be written as

$$\begin{aligned}
A_{+\frac{1}{2}}^{+\frac{1}{2}0} &= \left[\frac{C_R P_1^- + C_L P_1^+}{\sqrt{2}}\right] D_{+\frac{1}{2}}^{+\frac{1}{2}(\frac{1}{2})*}(\phi,\theta,0),\\
A_{+\frac{1}{2}}^{-\frac{1}{2}0} &= \left[\frac{C_R P_1^+ + C_L P_1^-}{\sqrt{2}}\right] D_{-\frac{1}{2}}^{+\frac{1}{2}(\frac{1}{2})*}(\phi,\theta,0),\\
A_{-\frac{1}{2}}^{+\frac{1}{2}0} &= \left[\frac{C_R P_1^- + C_L P_1^+}{\sqrt{2}}\right] D_{+\frac{1}{2}}^{-\frac{1}{2}(\frac{1}{2})*}(\phi,\theta,0),\\
A_{-\frac{1}{2}}^{-\frac{1}{2}0} &= \left[\frac{C_R P_1^+ + C_L P_1^-}{\sqrt{2}}\right] D_{-\frac{1}{2}}^{-\frac{1}{2}(\frac{1}{2})*}(\phi,\theta,0),
\end{aligned} \tag{86}$$

where

$$P_1^{\pm} = \sqrt{m_3}\frac{E_1 + m_1 \pm q}{\sqrt{E_1 + m_1}}, \qquad P_2^{\pm} = \frac{1}{\sqrt{2}}\sqrt{\frac{E_2 \pm q}{E_2 \mp q}}. \tag{87}$$

By comparing Eqs. (85) and (86), the helicity-coupling amplitude can be determined.

3. *Two-body decay amplitude in any frame*

In the above discussion, we have presented the two-body decay amplitudes in the two-body decay COM frame. In practice, however, the momenta of the particles in the decay process are generally not given in the COM frame but in any frame, such as the lab frame. Therefore, to evaluate the amplitude for a given event in the lab frame, one must boost the event back to the COM frame. Such a boost acts nontrivially on the final-state particles and induces a nontrivial Wigner rotation. Consequently, the amplitude in any frame can be written as the corresponding COM frame amplitude multiplied by the appropriate Wigner D-matrices for the final-state particles.

In any frame, within the traditional-LS partial wave expansion, the canonical amplitudes can be organized by the LS coupling amplitudes, the CGCs, the spherical harmonics in the COM frame, and the Wigner D-matrices induced by the boost. It can be written as

$$\begin{aligned}
A_{\sigma_3}^{\sigma_1\sigma_2}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2) = &\sum_{LS\sigma_L\sigma_S\sigma_1'\sigma_2'} g_{LS}\, C_{s_3,\sigma_3}^{L,\sigma_L;S,\sigma_S} C_{S,\sigma_S}^{s_1,\sigma_1';s_2,\sigma_2'} Y_{\sigma_L}^{L}(\Omega)\\
&\times D_{\sigma_1}^{\sigma_1'(s_1)*}(R_{31}) D_{\sigma_2}^{\sigma_2'(s_2)*}(R_{32}),
\end{aligned} \tag{88}$$

where R_{31} and R_{32} are the Wigner rotations induced on particle-1 and particle-2 by boosting the amplitude from any frame to the rest frame of particle-3 through the canonical-standard boost of particle-3. This Wigner rotation is exactly the canonical little group rotation in Eq. (29), and it can be written as

$$R_{3i} = W(L_{3,c}^{-1}(\mathbf{p}_3), p_1). \tag{89}$$

In contrast, the helicity amplitudes are organized by the helicity-coupling amplitudes, the helicity Wigner D-matrix in the COM frame, and the Wigner D-matrices induced by the boost. It can be expressed as

$$\begin{aligned}
&A_{\lambda_3}^{\lambda_1\lambda_2}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2)\\
&= \sum_{\lambda_1'\lambda_2'} \sqrt{\frac{2s_3+1}{4\pi}} D_{\lambda_1'-\lambda_2'}^{\lambda_3(s_3)*}(\phi,\theta,0) H^{\lambda_1'\lambda_2'} D_{\lambda_1}^{\lambda_1'(s_1)*}(R_{31}') D_{\lambda_2}^{\lambda_2'(s_2)*}(R_{32}'),
\end{aligned} \tag{90}$$

where R_{31}' and R_{32}' are the Wigner rotations induced on particle-1 and particle-2 by boosting the amplitude from any frame to the rest frame of particle-3 through the helicity standard boost of particle-3. This Wigner rotation is exactly the helicity little group rotation in Eq. (29), and it can be written as

$$R'_{3i} = W(L^{-1}_{3,h}(\mathbf{p}_3), p_1). \tag{91}$$

In what follows, we evaluate this Wigner rotation explicitly for both the canonical and helicity amplitudes, and provide the corresponding amplitude expressions in any frame.

Canonical amplitude in any frame In any frame, such as the lab frame, we define the canonical amplitude as

$$A^{\sigma_1\sigma_2}_{\sigma_3}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2) \equiv \langle \mathbf{p}_1\sigma_1;\mathbf{p}_2\sigma_2|M|\mathbf{p}_3\sigma_3\rangle, \tag{92}$$

where $\mathbf{p}_3$, $\mathbf{p}_1$, and $\mathbf{p}_2$ are the momenta in the lab frame. To calculate this amplitude, the system is boosted to the COM frame. In the canonical basis, the corresponding boost is $L^{-1}_{3,c}(p_3)$, where the subscript 3 indicates that this is the standard boost associated with particle-3.

Under the Lorentz transformation, the canonical amplitude becomes

$$\langle \mathbf{p}_1\sigma_1;\mathbf{p}_2\sigma_2|M|\mathbf{p}_3\sigma_3\rangle \xrightarrow{L^{-1}_{3,c}(p_3)} \langle \mathbf{p}^*_1\sigma'_1;\mathbf{p}^*_2\sigma'_2|M|\mathbf{k}_3\sigma_3\rangle, \tag{93}$$

where $\mathbf{k}_3$, $\mathbf{p}^*_1$, and $\mathbf{p}^*_2$ are the momenta of particle-3, particle-1, and particle-2 in the two-body decay COM frame.

For particle-3, the boost that brings it from the LAB frame to its rest frame is collinear with its momentum. According to Eq. (33), such a collinear boost induces a trivial little group element, so no additional rotation acts on the spin indices. In contrast, for particle-1 and particle-2, one obtains nontrivial little group rotations. The reason is that the transformation from the LAB frame to the rest frame of particle-i is not unique. One option is to reach the particle-i rest frame by a single pure boost. Alternatively, one may first boost to the rest frame of particle-3 and then boost to the rest frame of particle-i. Although both constructions end in the rest frame of particle-i and map p_i to the standard rest momentum k_i, they define different spatial axes in that rest frame. The two definitions are related by a Wigner rotation. Therefore, when the two-step construction is adopted, the resulting rest-frame state must be supplemented by the corresponding little group rotation. A simple diagrammatic description for the Wigner rotation is shown in Fig. 5.

From Eq. (31), the transformations of the individual states can be written as

$$U[L^{-1}_{3,c}(\mathbf{p}_3)]|\mathbf{p}_i\sigma_i\rangle = \sum_{\sigma'_i} D^{\sigma'_i(s_i)}_{\sigma_i}(R_{3i})|\mathbf{p}^*_i\sigma'_i\rangle \quad (i=1,2,3), \tag{94}$$

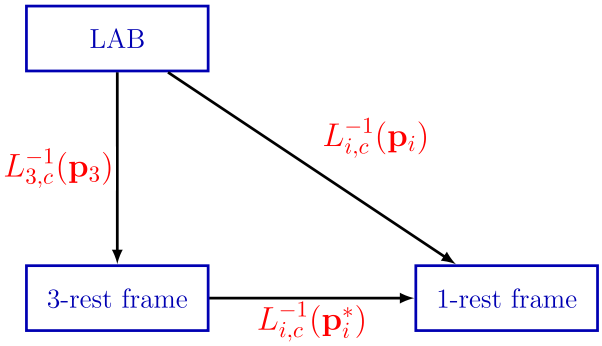

Fig. 5. (color online) Two boost paths from the lab frame to the rest frame of particle-i: a direct boost $L^{-1}_{i,c}(\mathbf{p}_i)$ and the two-step boost $L^{-1}_{i,c}(\mathbf{p}^*_i)L^{-1}_{3,c}(\mathbf{p}_3)$. The difference between these two paths is the Wigner rotation, which acts on the little group indices of particle-i.

where $\mathbf{p}^*_3 = \mathbf{k}_3$ and R_{3i} is the corresponding Wigner rotation, defined by[1)]

$$R_{3i} = L^{-1}_{i,c}(\mathbf{p}^*_i)L^{-1}_{3,c}(\mathbf{p}_3)L_{i,c}(\mathbf{p}_i) \quad (i=1,2,3), \tag{95}$$

where the subscript i indicates that this is the standard boost associated with particle-i. In particular, the Wigner rotation for particle-3 is trivial, $R_{33}=\mathbf{I}$. Therefore, only two nontrivial Wigner rotation representation matrices act on the amplitude. Explicitly, the transformed amplitude can be written as

$$\begin{aligned} &A^{\sigma_1\sigma_2}_{\sigma_3}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2) \\ &= \sum_{\sigma'_1\sigma'_2} A^{\sigma'_1\sigma'_2}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}^*_1,\mathbf{p}^*_2) D^{\sigma'_1(s_1)*}_{\sigma_1}(R_{31}) D^{\sigma'_2(s_2)*}_{\sigma_2}(R_{32}), \end{aligned} \tag{96}$$

where $A^{\sigma'_1\sigma'_2}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}^*_1,\mathbf{p}^*_2)$ are the amplitudes in the COM frame. With the above definitions in place, a detailed derivation of the Wigner rotation R_{3i} is presented below.

For a particle with four-momentum p, we introduce the velocity and the Lorentz factor as

$$\boldsymbol{\beta} = \frac{\mathbf{p}}{E}, \quad \beta = |\boldsymbol{\beta}|, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{97}$$

The canonical-standard boost of the particle is

$$L_c(\mathbf{p}) = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^T \\ \gamma\boldsymbol{\beta} & \mathbf{I} + \dfrac{\gamma-1}{\beta^2}\boldsymbol{\beta}\boldsymbol{\beta}^T \end{pmatrix}. \tag{98}$$

Its inverse is

1) The Wigner rotation is also known as the Wigner-Thomas rotation; see, e.g., sec. 6.7 in Ref. [82]

$$L_c^{-1}(\mathbf{p}) = L_c(-\mathbf{p}) = \begin{pmatrix} \gamma & -\gamma\beta^T \\ -\gamma\boldsymbol{\beta} & \mathbf{I}+\dfrac{\gamma-1}{\beta^2}\boldsymbol{\beta}\boldsymbol{\beta}^T \end{pmatrix}. \tag{99}$$

The composition of the first two Lorentz transformations of Eq. (95) is defined as

$$\Lambda \equiv L_{3,c}^{-1}(\mathbf{p}_3)L_{i,c}(\mathbf{p}_i) = L_{3,c}(-\mathbf{p}_3)L_{i,c}(\mathbf{p}_i). \tag{100}$$

In the rest frame of particle-i, the momentum of particle-i is the standard momentum $k_i^\mu = (m_i, \mathbf{0})^T$. The corresponding momentum p_i^* in the two-body decay COM frame is obtained by performing the aforementioned Lorentz transformation on k_i, namely,

$$p_i^* = \Lambda k_i. \tag{101}$$

Using Eqs. (97), (98), (99), (100), and (101), one can obtain

$$\gamma_i^* = \gamma_3\gamma_i(1-\boldsymbol{\beta}_3\cdot\boldsymbol{\beta}_i), \quad \boldsymbol{\beta}_i^* = \frac{-\gamma_3\boldsymbol{\beta}_3+\boldsymbol{\beta}_i+\dfrac{\gamma_3-1}{\beta_3^2}(\boldsymbol{\beta}_3\cdot\boldsymbol{\beta}_i)\boldsymbol{\beta}_3}{\gamma_3(1-\boldsymbol{\beta}_3\cdot\boldsymbol{\beta}_i)}. \tag{102}$$

Therefore, the Wigner rotation induced by the boost can be expressed as

$$R_{3i}(\omega_i) = L_{i,c}^{-1}(\mathbf{p}_i^*)L_{3,c}^{-1}(\mathbf{p}_3)L_{i,c}(\mathbf{p}_i), \tag{103}$$

where the $\boldsymbol{\omega}_i$ direction is parallel to $-\mathbf{p}_3\times\mathbf{p}_i$.

To obtain the explicit magnitude of the Wigner angle $\omega_i = |\boldsymbol{\omega}_i|$, we first rotate the coordinate system such that the z-axis is aligned with the unit vector

$$\hat{\mathbf{n}}_{3i} \equiv \frac{-\mathbf{p}_3\times\mathbf{p}_i}{|\mathbf{p}_3\times\mathbf{p}_i|}, \quad R_{\hat{\mathbf{n}}_{3i}}\hat{\mathbf{z}} = \hat{\mathbf{n}}_{3i}, \quad \cos\theta_{3i} = -\hat{\mathbf{p}}_3\cdot\hat{\mathbf{p}}_i, \tag{104}$$

where $R_{\hat{\mathbf{n}}_{3i}}$ is a spatial rotation that maps $\hat{\mathbf{z}}$ to $\hat{\mathbf{n}}_{3i}$. In the rotated frame, the two velocities lie in the x-y plane. We may choose coordinates such that

$$\boldsymbol{\beta}_3' = (\beta_3, 0, 0) \qquad \boldsymbol{\beta}_i' = (\beta_i\cos\theta_{3i}, \beta_i\sin\theta_{3i}, 0), \tag{105}$$

where θ_{3i} is the angle between $\boldsymbol{\beta}_3$ and $\boldsymbol{\beta}_i$ in the lab frame. In the rotated frame, one finds that the boosted velocity $\boldsymbol{\beta}_i^{*\prime}$ also satisfies $(\boldsymbol{\beta}_i^{*\prime})_z = 0$. Therefore, in this rotated frame, the Wigner rotation is a pure rotation about the z-axis:

$$R_{3i}' = R_z(\omega_{3i}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_{3i} & -\sin\omega_{3i} & 0 \\ 0 & \sin\omega_{3i} & \cos\omega_{3i} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{106}$$

A convenient form for ω_{3i} is obtained in terms of rapidities $\eta_a \equiv \operatorname{arctanh}\beta_a$ $(a = 3, i)$:

$$\tan\frac{\omega_{3i}}{2} = \frac{\sin\theta_{3i}\sinh\left(\dfrac{\eta_3}{2}\right)\sinh\left(\dfrac{\eta_i}{2}\right)}{\cosh\left(\dfrac{\eta_3}{2}\right)\cosh\left(\dfrac{\eta_i}{2}\right)+\cos\theta_{3i}\sinh\left(\dfrac{\eta_3}{2}\right)\sinh\left(\dfrac{\eta_i}{2}\right)}. \tag{107}$$

This determines the magnitude of the Wigner angle.

Finally, we rotate back to the original orientation. The Wigner rotation in the original frame can be expressed as

$$R_{3i}(\boldsymbol{\omega}_i) = R_{\hat{\mathbf{n}}}R_z(\omega_{3i})R_{\hat{\mathbf{n}}}^{-1}. \tag{108}$$

With R_{31} and R_{32} determined in this way, the canonical amplitude in any frame becomes

$$A_{\sigma_3}^{\sigma_1\sigma_2}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2) = \sum_{LS\sigma_L\sigma_S\sigma_1'\sigma_2'} g_{LS}C_{s_3,\sigma_3}^{L,\sigma_L;S,\sigma_S}C_{S,\sigma_S}^{s_1,\sigma_1';s_2,\sigma_2'}Y_{\sigma_L}^L(\Omega) \times D_{\sigma_1}^{\sigma_1'(s_1)*}(R_{31}(\boldsymbol{\omega}_1))D_{\sigma_2}^{\sigma_2'(s_2)*}(R_{32}(\boldsymbol{\omega}_2)). \tag{109}$$

Helicity amplitude in any frame For the helicity amplitude, the definition in any frame is

$$A_{\lambda_3}^{\lambda_1\lambda_2}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2) = \langle\mathbf{p}_1\lambda_1;\mathbf{p}_2\lambda_2|M|\mathbf{p}_3\lambda_3\rangle. \tag{110}$$

Similarly, the helicity amplitude is boosted to the COM frame. To preserve the helicity of the parent particle, the boost is chosen to be $L_h^{-1}(\mathbf{p}_3)$. However, the helicity boost is not a pure boost but a combination of a pure boost and a rotation. This is because, in the rest frame of particle-3, its spin quantization axis must point along the z-axis, while originally in the lab frame, its spin quantization axis points along the direction of its momentum. Hence, we need to rotate the z-axis to align with the momentum direction of particle-3. This implies that boosting back from the lab frame to the helicity COM frame does not yield the same frame as the canonical COM frame. Consequently, the momenta of particle-1 and particle-2 are different in these two COM frames. We define the momenta in the helicity COM frame as $\mathbf{q}_1^*$, $\mathbf{q}_2^*$, whose relation with the momenta $\mathbf{p}_1^*$, $\mathbf{p}_2^*$ in the canonical COM frame is given by

$$\mathbf{q}_i^* = L_{3,h}^{-1}(\mathbf{p}_3)\mathbf{p}_i = R^{-1}(\hat{\mathbf{p}}_3)L_{3,c}^{-1}(\mathbf{p}_3)\mathbf{p}_i = R^{-1}(\hat{\mathbf{p}}_3)\mathbf{p}_i^* \quad (i=1,2). \tag{111}$$

The relations among the three coordinate systems are illustrated in Fig. 6.

Under this Lorentz transformation, the helicity amplitude becomes

$$\begin{aligned}\langle \mathbf{p}_1\lambda_1;\mathbf{p}_2\lambda_2|M|\mathbf{p}_3\lambda_3\rangle &\xrightarrow{L_{3,c}^{-1}(\mathbf{p}_3)} \langle \mathbf{p}_1^*\tilde{\lambda}_1;\mathbf{p}_2^*\tilde{\lambda}_2|M|\mathbf{k}_3\lambda_3\rangle \\ &\xrightarrow{R^{-1}(\hat{\mathbf{p}}_3)} \langle \mathbf{q}_1^*\lambda_1';\mathbf{q}_2^*\lambda_2'|M|\mathbf{k}_3\lambda_3\rangle,\end{aligned} \tag{112}$$

where $\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3$ are the momenta defined in the lab frame with respect to the coordinate axes $\{x,y,z\}$, and the helicities $\lambda_1,\lambda_2,\lambda_3$ correspond to the spin projections along their respective momentum directions. In the COM frame, $\mathbf{k}_3,\mathbf{q}_1^*,\mathbf{q}_2^*$ are the momenta defined in the helicity COM frame with respect to the coordinate axes $\{x',y',z'\}$, and $\lambda_1',\lambda_2',\lambda_3$ again correspond to the spin projections along their respective momentum directions. In particular, since the direction of $\mathbf{p}_3$ in the lab frame corresponds to the z'-axis direction in the helicity COM frame, the helicity λ_3 is defined consistently in both frames.

Therefore, due to the helicity mismatch between the two reference frames, the helicity Wigner rotations must exist between the helicity particle states defined in these frames to align their helicities. Applying the boost to the spin states yields

$$\begin{aligned}U[L_{3,h}^{-1}(\mathbf{p}_3)]|\mathbf{p}_i\lambda_i\rangle &= L_{3,h}^{-1}(\mathbf{p}_3)L_{i,h}(\mathbf{p}_i)|\mathbf{k}_i\lambda_i\rangle \\ &= L_{i,h}(\mathbf{q}_i^*)L_{i,h}^{-1}(\mathbf{q}_i^*)L_{3,h}^{-1}(\mathbf{p}_3)L_{i,h}(\mathbf{p}_i)|\mathbf{k}_i\lambda_i\rangle \\ &= \sum_{\lambda_i'} D_{\lambda_i}^{\lambda_i'(s_i)}(R_{3i}')|\mathbf{q}_i^*\lambda_i'\rangle \quad (i=1,2),\end{aligned} \tag{113}$$

where R_{3i}' is the corresponding Wigner rotation defined as

$$R_{3i}' = L_{i,h}^{-1}(\mathbf{q}_i^*)L_{3,h}^{-1}(\mathbf{p}_3)L_{i,h}(\mathbf{p}_i) \quad (i=1,2). \tag{114}$$

The helicity amplitude in the lab frame can be expressed as

$$A_{\lambda_3}^{\lambda_1\lambda_2}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2) = \sum_{\lambda_1'\lambda_2'} A_{\lambda_3}^{\lambda_1'\lambda_2'}(\mathbf{k}_3,\mathbf{q}_1^*,\mathbf{q}_2^*)D_{\lambda_1}^{\lambda_1'(s_1)*}(R_{31}')D_{\lambda_2}^{\lambda_2'(s_2)*}(R_{32}'). \tag{115}$$

To factorize R_{3i}', we start with the definition of the helicity boost,

$$L_{i,h}^{-1}(\mathbf{q}_i^*) = R^{-1}(\mathbf{q}_i^*)L_{i,c}^{-1}(\mathbf{q}_i^*), \tag{116}$$

$$L_{3,h}^{-1}(\mathbf{p}_3) = R^{-1}(\mathbf{p}_3)L_{3,c}^{-1}(\mathbf{p}_3), \tag{117}$$

$$L_{i,h}(\mathbf{p}_i) = L_{i,c}(\mathbf{p}_i)R(\mathbf{p}_i). \tag{118}$$

Substituting these relations into the definition of R_{3i}' gives

$$R_{3i}' = R^{-1}(\mathbf{q}_i^*)\left(L_{i,c}^{-1}(\mathbf{q}_i^*)R^{-1}(\mathbf{p}_3)L_{3,c}^{-1}(\mathbf{p}_3)L_{i,c}(\mathbf{p}_i)\right)R(\mathbf{p}_i). \tag{119}$$

Using the property $RL_c^{-1}(\mathbf{p})R^{-1} = L_c^{-1}(R\mathbf{p})$ and Eq. (111), we have

$$L_{i,c}^{-1}(\mathbf{p}') \equiv R(\mathbf{p}_3)L_{i,c}^{-1}(\mathbf{q}_i^*)R^{-1}(\mathbf{p}_3), \qquad \mathbf{p}' = R(\mathbf{p}_3)\mathbf{q}_i^* = \mathbf{p}_i^*. \tag{120}$$

Therefore, by inserting $R^{-1}(\mathbf{p}_3)R(\mathbf{p}_3)$ into Eq. (119), we obtain

$$\begin{aligned}R_{3i}' &= R^{-1}(\mathbf{q}_i^*)R^{-1}(\mathbf{p}_3)\left(R(\mathbf{p}_3)L_{i,c}^{-1}(\mathbf{q}_i^*)R^{-1}(\mathbf{p}_3)L_{3,c}^{-1}(\mathbf{p}_3)L_{i,c}(\mathbf{p}_i)\right)R(\mathbf{p}_i) \\ &= R^{-1}(\mathbf{q}_i^*)R^{-1}(\hat{\mathbf{p}}_3)\left(L_{i,c}^{-1}(\mathbf{p}_i^*)L_{3,c}^{-1}(\mathbf{p}_3)L_{i,c}(\mathbf{p}_i)\right)R(\mathbf{p}_i).\end{aligned} \tag{121}$$

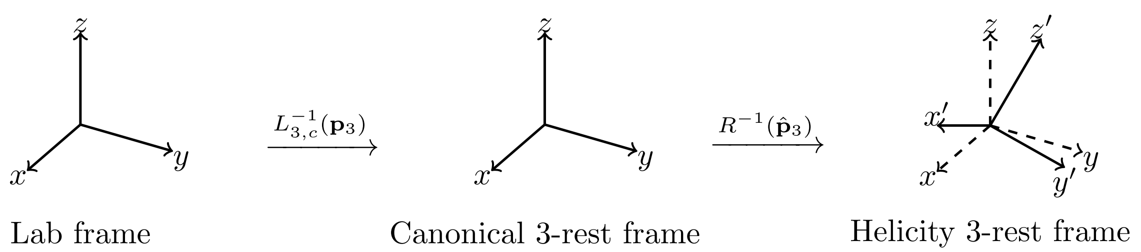

Fig. 6. The figure illustrates the relations among the three coordinate systems. The lab frame and the canonical 3-rest frame are connected by a pure boost, which does not change the orientation of the axes; therefore, they share the same set of spatial axes x,y,z. For the helicity 3-rest frame, an additional rotation is applied, resulting in a change in the spatial axes. The corresponding new axes are denoted as x',y',z'.

It can be observed that the expression inside the bracket corresponds to the Wigner rotation in the canonical form given by Eq. (108). Therefore, the final Wigner rotation for the helicity state is given by

$$R'_{3i} = R^{-1}(\mathbf{q}_i^*)R^{-1}(\mathbf{p}_3)R_{3i}(\omega_i)R(\mathbf{p}_i). \tag{122}$$

Finally, the amplitude in any frame can be expressed as

$$\begin{aligned} &A_{\lambda_3}^{\lambda_1\lambda_2}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2) \\ &= \sum_{\lambda'_1\lambda'_2} \sqrt{\frac{2s_3+1}{4\pi}} D_{\lambda'_1-\lambda'_2}^{\lambda_3(s_3)*}(\phi,\theta,0) H^{\lambda'_1\lambda'_2} D_{\lambda_1}^{\lambda'_1(s_1)*}(R'_{31}) D_{\lambda_2}^{\lambda'_2(s_2)*}(R'_{32}). \end{aligned} \tag{123}$$

III. COVARIANT ZEMACH AND PROJECTION TENSOR AMPLITUDES

In this section, tensor constructions of LS amplitudes are discussed. This approach can be traced back to the Zemach tensor method [21]. In the Zemach tensor method, two-body decays are analyzed in the parent rest frame. Unlike the traditional-LS coupling discussed in Section II, where angular momenta are coupled in the little group representation using CGCs, the Zemach tensor method first builds angular momentum tensors associated with the orbital and spin parts and then performs the required couplings and projections at the tensor level using the invariant tensors of $SO(3)$ and $SU(2)$.

However, the Zemach tensor method is not covariant. To construct covariant partial wave amplitudes, the covariant tensor method is introduced [32, 33]. Its key idea is to construct covariant coupling structures and obtain the corresponding partial wave amplitudes by fully contracting them with the external spin wave functions over the Lorentz indices. This idea has been developed into a more systematic framework [48, 49]. Depending on the choice of external spin wave functions, two schemes are commonly used.

The first scheme, referred to as the PS scheme, returns to the COM frame and defines the final-state spin wave functions at the standard momenta. In this construction, the amplitude can be written as

$$A(k_3,p_1^*,p_2^*;L,S) = \underbrace{\Gamma(k_3,p_1^*,p_2^*;L,S)}_{\text{covariant } LS \text{ coupling structure}} \times \underbrace{U(k_3,k_1,k_2;s_3,s_1,s_2)}_{\text{pure-spin part}}. \tag{124}$$

In the COM frame, this scheme allows for a complete separation between orbital and spin. Consequently, it is equivalent to the traditional-LS method. By applying a Lorentz boost, this definition can be extended to any frame and evaluated therein. However, the boost induces Wigner rotations, and these rotations depend on the COM frame.

The second scheme is referred to as the GS scheme, which can be written as

$$C(p_3,p_1,p_2;L,S) = \underbrace{\Gamma(p_3,p_1,p_2;L,S)}_{\text{covariant } LS \text{ coupling structure}} \times \underbrace{U(p_3,p_1,p_2;s_3,s_1,s_2)}_{\text{general-spin part}}. \tag{125}$$

In this construction, the amplitude is covariant, and no additional boost back to the COM frame is required. This makes the treatment in any frame more natural. However, from the structure of the amplitude, one can directly see that, in this scheme, orbital angular momentum and spin angular momentum cannot be completely separated. Part of the orbital information is mixed into the spin part. As a result, this method is not fully equivalent to the traditional-LS method. Instead, it defines an alternative LS partial wave basis.

In subsection III.A, we review the Zemach tensor method and then introduce the covariant tensor method, clarifying the correspondence between the two constructions. In subsection III.B, we discuss the covariant projection tensor method both in the PS scheme and the GS scheme. In subsection III.C, we provide several explicit calculation examples for different LS amplitudes.

A. Covariant tensor amplitudes

In this subsection, the tensor method for constructing the LS amplitudes is introduced. Its key idea is to formulate angular momentum coupling in tensor representations so that the coupling is implemented by index contractions rather than by explicit CGCs in the little group.

The discussion starts from the non-relativistic Zemach tensor method [21]. In this method, each spin is encoded in an appropriate tensor basis, and the angular momentum coupling is implemented by direct contractions of tensor indices.

A covariant extension is then obtained by lifting the construction to Lorentz tensors. In the covariant tensor method [32, 33], the amplitude is built by contracting Lorentz indices in $SO(3,1)$ representations. This construction is manifestly covariant, while the separation between the orbital and spin parts is not explicit.

In subsubsection III.A.1, the construction of the Zemach tensor method is discussed, while in subsubsection III.A.2, the construction of the covariant tensor method is discussed.

1. *Zemach tensor method*

The non-relativistic Zemach tensor method [21] formulates the angular momentum coupling in tensor representations. In this formulation, the coupling is implemented by contracting tensor indices with the invariant tensors of $SO(3)$ and $SU(2)$.

For contractions between pure $SO(3)$ tensors, the corresponding invariant contraction tensors are ε_{ijk} and δ_{ij}. For contractions involving both $SU(2)$ and $SO(3)$ tensors, the corresponding invariant contraction tensors are $\varepsilon_{\alpha\beta}$, δ^{α}_{β}, and $(\sigma_i)^{\alpha}_{\beta}$. Spatial indices $i,j,k=1,2,3$ label the Cartesian basis of the $SO(3)$ vector representation, while spinor indices $\alpha,\beta=1,2$ label the fundamental representation of $SU(2)$. With these conventions fixed, contractions with ε_{ijk}, δ_{ij}, $\varepsilon_{\alpha\beta}$, δ^{α}_{β}, and $(\sigma_i)^{\alpha}_{\beta}$ are equivalent to the CGCs used to couple angular momenta in the little group formulation in Eq. (70).

Therefore, in the non-relativistic Zemach tensor method, the orbital-spin coupling in a two-body decay amplitude is realized by constructing the orbital tensor and the spin tensors for each external particle, and then contracting them with the $SO(3)$ or $SU(2)$ invariant tensors. The following discussion addresses the constructions of the spin tensors and the orbital tensors in turn.

Let us first consider the pure orbital part. In the two-body decay COM frame, the final-state three-momenta are denoted by $\mathbf{p}_1^*$ and $\mathbf{p}_2^*$, and the relative momentum is defined as

$$\mathbf{r}=\mathbf{p}_1^*-\mathbf{p}_2^*. \tag{126}$$

In Section II, the construction of the orbital angular momentum L in the LS coupling is given in Eq. (44). The pure orbital part is described by the spherical harmonics associated with the direction of the relative momentum, and these spherical harmonics correspond to the pure-L component in the $SO(3)$ little group representation.

Therefore, when constructing the tensor structure for the orbital part, the dependence is taken only on the relative momentum, and an appropriate projection is applied to extract the pure-L rank-L tensor structure. This projection is equivalent to taking the symmetric traceless part of the rank-L tensor formed directly from the tensor product of the relative momentum. The resulting symmetric traceless rank-L tensor furnishes the irreducible $SO(3)$ representation with orbital angular momentum L. Consequently, the spherical harmonics admit an equivalent expression in the tensor representation as

$$Y^L_{\sigma_L}(\Omega)\simeq T^L_{i_1\cdots i_L}(\mathbf{r}\cdots\mathbf{r})\equiv \mathbf{r}_{i_1}\cdots\mathbf{r}_{i_L}-\text{trace part}, \tag{127}$$

where $i_1,\ldots,i_L$ are spatial indices. The relation above holds up to an overall normalization factor, which is omitted for simplicity. Here, $T^L_{i_1\cdots i_L}(\mathbf{r}\cdots\mathbf{r})$ denotes the **symmetric traceless** rank-L tensor constructed from the relative momentum. It satisfies the symmetry condition

$$T^L_{i_1\cdots i_a\cdots i_b\cdots i_L}=T^L_{i_1\cdots i_b\cdots i_a\cdots i_L},\qquad \forall a,b, \tag{128}$$

and the tracelessness condition

$$\sum_{i_a i_b}\delta_{i_a i_b}T^L_{i_1\cdots i_a\cdots i_b\cdots i_L}=0,\qquad \forall a\neq b. \tag{129}$$

For low values of L, the explicit expressions are

$$\begin{aligned}
&T^0=1,\\
&T^1_i(\mathbf{r})=\mathbf{r}_i,\\
&T^2_{ij}(\mathbf{rr})=\mathbf{r}_i\mathbf{r}_j-\frac{1}{3}r^2\delta_{ij},\\
&T^3_{ijk}(\mathbf{rrr})=\mathbf{r}_i\mathbf{r}_j\mathbf{r}_k-\frac{1}{5}\mathbf{r}^2(\delta_{ij}\mathbf{r}_k+\delta_{jk}\mathbf{r}_i+\delta_{ki}\mathbf{r}_j).
\end{aligned} \tag{130}$$

Next, the spin part is discussed. For the spin part, it is first necessary to describe particles in the tensor formulation. According to the value of the particle spin, there are two classes: integer spin and half-integer spin. Since higher-spin tensors can be obtained by angular momentum coupling of lower-spin objects, only two basic building blocks are required: the spin-1 and spin-1/2 tensors. In the non-relativistic formulation, they are equivalent to the spin wave functions defined in the particle rest frame, where the spin-1 building block is the three-dimensional polarization vector $(\epsilon_\sigma)_i$, and the spin-$\frac{1}{2}$ building block is the two-component Pauli spinor $(\chi_\sigma)^\alpha$. By taking tensor products of these basic objects and projecting onto the irreducible representation with the desired spin, one obtains the spin tensors required for the coupling of the particles in the pure spin part.

1. For integer spin, the basic building block is the three-dimensional polarization vector defined in the rest frame. For initial-state particles, the polarization vector is denoted by $(\epsilon_\sigma)_i$ and can be written as

$$\begin{aligned}
\epsilon_+&=-\frac{1}{\sqrt{2}}(1,\quad \mathrm{i},\quad 0),\\
\epsilon_-&=\frac{1}{\sqrt{2}}(1,\quad -\mathrm{i},\quad 0),\\
\epsilon_0&=(0,\quad 0,\quad 1),
\end{aligned} \tag{131}$$

where the subscripts $+$, $-$, and 0 indicate the allowed values of the polarization label σ in $(\epsilon_\sigma)_i$, the symbol i appearing in the vector components denotes the imaginary unit, and the index $i=1,2,3$ labels Cartesian $SO(3)$ components. The complex conjugation of these polarization vectors gives the conjugate vector $(\epsilon^{*\sigma})_i$, which is used for final-state particles:

$$(\epsilon_\sigma)_i^* = (\epsilon^{*\sigma})_i. \tag{132}$$

The polarization vectors in Eq. (131) are then used as the building blocks for the integer spin-s tensor.

In the non-relativistic formulation, a spin-s particle is represented by a rank-s symmetric traceless tensor $(T^s_\sigma)_{i_1\cdots i_s}$, where the label σ denotes the spin projection of spin-s and takes values $\sigma=-s,\cdots,s$. This label is distinct from the spin-1 polarization label used in $(\epsilon_\sigma)_i$.

The symmetry and tracelessness conditions ensure that $(T^s_\sigma)_{i_1\cdots i_s}$ furnishes the irreducible $SO(3)$ representation with spin-s. For an initial-state particle, the spin tensor is taken as $(T^s_\sigma)_{i_1\cdots i_s}$, while for a final-state particle, the corresponding object is obtained by complex conjugation, which can be defined as

$$\left(T^s_\sigma\right)^*_{i_1\cdots i_s} \equiv \left(T^{*s,\sigma}\right)_{i_1\cdots i_s}. \tag{133}$$

To construct the spin-s tensor $(T^s_\sigma)_{i_1\cdots i_s}$ explicitly, one may start with the s-fold tensor product of polarization vectors and project onto the symmetric traceless part. It can be defined as

$$(T^s_{\sigma_1\cdots\sigma_s})_{i_1\cdots i_s} \equiv \underbrace{(\epsilon_{\sigma_1})_{i_1}\cdots(\epsilon_{\sigma_s})_{i_s}}_{\text{symmetrized in } i_1,\cdots,i_s} - \text{trace part}, \tag{134}$$

where each label $\sigma_r=+1,0,-1$ $(r=1,\cdots,s)$ specifies the polarization choice of the rth spin-1 building block $(\epsilon_{\sigma_r})_{i_r}$. For example, some low-spin tensors can be written explicitly as

$$\begin{aligned}
&T^0 = 1,\\
&(T^1_\sigma)_i = (\epsilon_\sigma)_i,\\
&(T^2_{\sigma_1\sigma_2})_{ij} = \frac{1}{2}\Big((\epsilon_{\sigma_1})_i(\epsilon_{\sigma_2})_j + (\epsilon_{\sigma_1})_j(\epsilon_{\sigma_2})_i\Big)\\
&\qquad -\frac{1}{3}(\epsilon_{\sigma_1}\cdot\epsilon_{\sigma_2})\delta_{ij},\\
&(T^3_{\sigma_1\sigma_2\sigma_3})_{ijk} = \frac{1}{3!}\Big((\epsilon_{\sigma_1})_i(\epsilon_{\sigma_2})_j(\epsilon_{\sigma_3})_k + \{ijk\}\Big)\\
&\qquad -\frac{1}{5}\Big(\delta_{ij}V_k + \delta_{jk}V_i + \delta_{ki}V_j\Big),\\
&\text{with}\quad V_k = \frac{1}{3}\Big((\epsilon_{\sigma_1}\cdot\epsilon_{\sigma_2})(\epsilon_{\sigma_3})_k + (\epsilon_{\sigma_3}\cdot\epsilon_{\sigma_1})(\epsilon_{\sigma_2})_k\\
&\qquad + (\epsilon_{\sigma_2}\cdot\epsilon_{\sigma_3})(\epsilon_{\sigma_1})_k\Big),
\end{aligned} \tag{135}$$

where $\{ijk\}$ denotes the set of all 3! permutations of the indices i,j,k. After performing the symmetric traceless projection on the vector indices in Eq. (134), and with the chosen normalization convention, the multi-label σ_r can be reorganized into a single spin-projection label $\sigma=-s,\cdots,s$. For convenience, the normalized components are chosen according to the following convention. For $0\le\sigma\le s$, define

$$(T^s_\sigma)_{i_1\cdots i_s} \equiv \sqrt{\frac{(2s)!}{2^{s-\sigma}(s+\sigma)!(s-\sigma)!}}(T^s_{\underbrace{0\cdots0}_{s-\sigma}\underbrace{1\cdots1}_{\sigma}})_{i_1\cdots i_s}, \tag{136}$$

where the multi-label subscript on the tensor on the right-hand side specifies the component such that, among the s polarization labels, σ of them are equal to $+1$, and the remaining $s-\sigma$ are equal to 0. Additionally, the symbol 1 in the subscript denotes the value $+1$ of the spin-1 polarization label. For $0\le\sigma\le s$, define the negative projection components by

$$(T^s_{-\sigma})_{i_1\cdots i_s} \equiv (-1)^\sigma (T^s_\sigma)^*_{i_1\cdots i_s}. \tag{137}$$

2. For half-integer spin, the basic building block is the two-component Pauli spinor defined in the rest frame. For initial-state particles, the spinor is denoted by $(\chi_\sigma)^\alpha$ and can be written as

$$\chi_+ = \begin{pmatrix}1\\0\end{pmatrix}, \chi_- = \begin{pmatrix}0\\1\end{pmatrix}, \tag{138}$$

where the subscripts + and − correspond to the spin projections $\sigma=+\frac{1}{2},-\frac{1}{2}$ of $(\chi_\sigma)^\alpha$. The complex conjugation of these spinors gives the conjugate spinor $(\chi^{*\sigma})_\alpha$, which is used for final-state particles:

$$(\chi_\sigma)^{\alpha*} = (\chi^{*\sigma})_\alpha. \tag{139}$$

In the non-relativistic formulation, a half-integer spin-$(s+\frac{1}{2})$ initial-state particle is described by a mixed tensor $(T^{s+\frac{1}{2}}_\sigma)^\alpha_{i_1\cdots i_s}$, which carries s vector indices $i_1,\ldots,i_s$ and one spinor index α. To extract the irreducible spin-$(s+\frac{1}{2})$ component from the reducible product space of a rank-s symmetric traceless tensor and a spinor, two essential conditions must be imposed. First, the tensor must be completely symmetric and traceless in all its vector indices. Second, it must satisfy an additional constraint, namely that its contraction with the Pauli matrices vanishes, which can be written as

$$\sum_{i_a}(\sigma_{i_a})^{\alpha}_{\beta}(T^{s+\frac{1}{2}}_{\sigma})^{\beta}_{i_1\cdots i_a\cdots i_s}=0,\qquad 1\le a\le s,\tag{140}$$

where $(\sigma_{i_a})^{\alpha}_{\beta}$ are the Pauli matrices. The role of this condition is to eliminate the lower spin-$(s-\frac{1}{2})$ components contained in the direct product space $T^s\otimes\chi$, thereby ensuring that the tensor describes only the pure spin-$(s+\frac{1}{2})$ representation.

An explicit construction that realizes this projection for an initial-state particle is provided by the following expression, which can be defined as

$$\begin{aligned}(T^{s+\frac{1}{2}}_{\sigma_s\sigma'})^{\alpha}_{i_1\cdots i_s}\equiv{}&(T^s_{\sigma_s})_{i_1\cdots i_s}(\chi_{\sigma'})^{\alpha}\\&+\frac{\mathrm{i}}{s+1}\sum_{a=1}^{s}\sum_{jk}\varepsilon_{i_a jk}(\sigma_j)^{\alpha}_{\beta}(T^s_{\sigma_s})_{i_1\cdots i_{a-1}k\,i_{a+1}\cdots i_s}(\chi_{\sigma'})^{\beta},\end{aligned}\tag{141}$$

where $\varepsilon_{i_a jk}$ is the totally antisymmetric tensor, the labels $\sigma_s=-s,\cdots,s$ and $\sigma'=-\frac{1}{2},\frac{1}{2}$ in the subscripts of $T^s_{\sigma_s}$ and $\chi_{\sigma'}$ denote spin projection quantum numbers, the symbol $(\sigma_j)^{\alpha}_{\beta}$ denotes the Pauli matrices, and the symbol i in the coefficient denotes the imaginary unit. The two labels σ_s and σ' can be combined into a single total label $\sigma=-s-\frac{1}{2},\cdots,s+\frac{1}{2}$ by imposing an appropriate normalization convention, which can be written as

$$\begin{aligned}\left(T_{\sigma}^{s+\frac{1}{2}}\right)^{\alpha}_{i_1\cdots i_s}={}&\sqrt{\frac{s+\sigma+\frac{1}{2}}{2s+1}}\left(T^{s+\frac{1}{2}}_{\sigma-\frac{1}{2},+\frac{1}{2}}\right)^{\alpha}_{i_1\cdots i_s}\\&+\sqrt{\frac{s-\sigma+\frac{1}{2}}{2s+1}}\left(T^{s+\frac{1}{2}}_{\sigma+\frac{1}{2},-\frac{1}{2}}\right)^{\alpha}_{i_1\cdots i_s},\end{aligned}\tag{142}$$

where the two tensors on the right-hand side are the tensors defined in Eq. (141), and the subscripts $(\sigma-\frac{1}{2},\frac{1}{2})$ and $(\sigma+\frac{1}{2},-\frac{1}{2})$ correspond to (σ_s,σ'), respectively. With this construction, the resulting tensor satisfies both the symmetry and tracelessness conditions in the vector indices and the constraint in Eq. (140). The corresponding final-state spin tensor can be obtained by taking the complex conjugate of the spin tensor in Eq. (142), which is defined by complex conjugation as

$$\left(T_{\sigma}^{s+\frac{1}{2}}\right)^{*\alpha}_{i_1\cdots i_s}\equiv\left(T^{*s+\frac{1}{2},\sigma}\right)_{i_1\cdots i_s\alpha}.\tag{143}$$

After constructing the basic tensors above, the amplitude can be obtained by contracting the tensor indices. The symbol $\otimes$ is used to denote angular momentum coupling in a unified way. More specifically, if two tensors carry angular momenta j_1 and j_2, their coupling is written as $T^{j_1}\otimes T^{j_2}$, meaning that the product representation is decomposed into irreducible components with total angular momentum J as

$$T^{j_1}\otimes T^{j_2}=\bigoplus_{J=|j_1-j_2|}^{j_1+j_2}T^J.\tag{144}$$

In the Zemach tensor method, this coupling is implemented by the invariant tensors of $SO(3)$ and $SU(2)$. Equivalently, it corresponds to performing the angular momentum coupling and projection in the little group representation using the CGCs. In what follows, several explicit examples of spin coupling are presented, which can be generalized to more general angular momentum couplings.

For the coupling of two integer spins s_1 and s_2, the coupling to a total spin-S tensor can be denoted by

$$T^{*s_1}\otimes T^{*s_2}\to T^S,\tag{145}$$

where the allowed values of S satisfy $|s_1-s_2|\le S\le s_1+s_2$. In particular, for $S=s_1+s_2$, the highest-spin tensor is obtained by taking the direct tensor product of the two spin tensors and then projecting onto the symmetric traceless part. The resulting total spin-S tensor can be written as

$$\begin{aligned}(T^{S=s_1+s_2,\sigma_1\sigma_2})_{i_1\cdots i_{s_1}j_1\cdots j_{s_2}}={}&\underbrace{(T^{*s_1,\sigma_1})_{i_1\cdots i_{s_1}}(T^{*s_2,\sigma_2})_{j_1\cdots j_{s_2}}}_{\text{symmetrized in all vector indices}}\\&-\text{trace part}.\end{aligned}\tag{146}$$

To obtain lower-spin components with $S=s_1+s_2-2k$ $(k=1,2,\cdots)$, k pairs of indices between the two tensors are contracted using δ_{ij}. The remaining spin components of the form $S=s_1+s_2-2k-1$ can be constructed from the rank-(s_1+s_2-2k) tensor by using the antisymmetric tensor ε_{ijk} to contract one additional index. As an explicit example, for the coupling of two spin-1 particles, $s_1=s_2=1$, the possible total spins are $S=0,1,2$, and the corresponding total spin tensors can be written as

$$\begin{aligned}T^{S=0,\sigma_1\sigma_2}={}&\sum_{ij}(T^{*1,\sigma_1})_i(T^{*1,\sigma_2})_j\delta_{ij},\\(T^{S=1,\sigma_1\sigma_2})_i={}&\sum_{jk}\varepsilon_{ijk}(T^{*1,\sigma_1})_j(T^{*1,\sigma_2})_k,\\(T^{S=2,\sigma_1\sigma_2})_{ij}={}&\frac{1}{2}\left[(T^{*1,\sigma_1})_i(T^{1,\sigma_2})_j+(T^{*1,\sigma_1})_j(T^{*1,\sigma_2})_i\right]\\&-\frac{1}{3}\delta_{ij}T^{S=0,\sigma_1\sigma_2}.\end{aligned}\tag{147}$$

It can be seen that the coupling between different angular momenta does not require the explicit evaluation of CGCs, as it is achieved by contracting tensor indices in the corresponding tensor representations.

For the coupling of two half-integer spins $s_1+\frac{1}{2}$ and $s_2+\frac{1}{2}$, the coupling to a total spin-S tensor can be denoted by

$$T^{*s_1+\frac{1}{2}}\otimes T^{*s_2+\frac{1}{2}}\to T^S, \tag{148}$$

where the allowed values of S satisfy $|s_1-s_2|\le S\le s_1+s_2+1$. The highest-spin tensor with $S=s_1+s_2+1$ is obtained by using the tensor $(\sigma_k)^\alpha_\beta$ to contract the two spinor indices into the vector index k, which results in a rank-(s_1+s_2+1) tensor. It can be written as

$$\begin{aligned}&(T^{S=s_1+s_2+1,\sigma_1\sigma_2})_{i_1\cdots i_{s_1}j_1\cdots j_{s_2}k}\\ &=\underbrace{(\sigma_k)^\alpha_\gamma\varepsilon^{\gamma\beta}(T^{*s_1+\frac{1}{2},\sigma_1})_{i_1\cdots i_{s_1}\alpha}(T^{*s_2+\frac{1}{2},\sigma_2})_{j_1\cdots j_{s_2}\beta}}_{\text{symmetrized in all vector indices}}-\text{trace part.}\end{aligned} \tag{149}$$

The remaining spin part is obtained by contracting with invariant tensors such as $\varepsilon_{\alpha\beta}$, δ^α_β, δ_{ij}, and ε_{ijk}, followed by taking the symmetric traceless part. As an example, for the coupling of spin-$\frac{1}{2}$ and spin-$\frac{1}{2}$, the possible total spins are $S=0$ and $S=1$. The corresponding tensors can be written as

$$\begin{aligned}T^{S=0,\sigma_1\sigma_2}&=\varepsilon^{\alpha\beta}(T^{*\frac{1}{2},\sigma_1})_\alpha(T^{*\frac{1}{2},\sigma_2})_\beta,\\ (T^{S=1,\sigma_1\sigma_2})_i&=(\sigma_i)^\alpha_\gamma\varepsilon^{\gamma\beta}(T^{*\frac{1}{2},\sigma_1})_\alpha(T^{*\frac{1}{2},\sigma_2})_\beta.\end{aligned} \tag{150}$$

For the coupling of an integer spin-s_1 and a half-integer spin-$s_2+\frac{1}{2}$, the coupling to a total spin-S tensor can be denoted by

$$T^{*s_1}\otimes T^{*s_2+\frac{1}{2}}\to T^S, \tag{151}$$

where the allowed values of S satisfy $\left|s_1-s_2-\frac{1}{2}\right|\le S\le s_1+s_2+\frac{1}{2}$. For $s_1\le s_2$, one can contract indices in the same way as for two integer spin tensors. For $s_1>s_2$, the lowest total spin component $S=s_1-s_2-\frac{1}{2}$ requires the use of the tensor $(\sigma_i)^\alpha_\beta$ to contract the spinor index with one of the vector indices. As an example, for the coupling of spin-1 and spin-$\frac{1}{2}$, the possible total spins are $S=\frac{1}{2}$ and $S=\frac{3}{2}$. The corresponding tensors can be written as

$$\begin{aligned}(T^{S=\frac{1}{2},\sigma_1\sigma_2})_\alpha&=\sum_i(\sigma_i)^\beta_\alpha(T^{*1,\sigma_1})_i(T^{*\frac{1}{2},\sigma_2})_\beta,\\ (T^{S=\frac{3}{2},\sigma_1\sigma_2})_{i\alpha}&=(T^{*1,\sigma_1})_i(T^{*\frac{1}{2},\sigma_2})_\alpha-\tfrac{1}{3}(\sigma_i)^\beta_\alpha(T^{S=\frac{1}{2},\sigma_1\sigma_2})_\beta.\end{aligned} \tag{152}$$

Having constructed the spin tensors and their coupling relations above, they can be extended to general angular momentum couplings and used to build the corresponding LS partial wave amplitudes. Once the final-state spins have been coupled into a total spin-S tensor, the next step is to couple this tensor with the orbital angular momentum L tensor to obtain a tensor structure with total angular momentum s_3, namely

$$T^S\otimes T^L\to T^{s_3}, \tag{153}$$

where T^{s_3} denotes the rank-s_3 tensor obtained from the LS coupling. Finally, this tensor is contracted with the external spin tensor of the parent particle with spin s_3. Once all indices have been contracted, the LS amplitude is obtained as

$$\begin{aligned}&A^{\sigma_1\sigma_2}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}_1^*,\mathbf{p}_2^*;L,S)\\ &=\underbrace{T^{s_3}_{\sigma_3}\cdot\Big[(T^{*s_1\sigma_1}\otimes T^{*s_2\sigma_2})^S}_{\text{spin part}}\otimes\underbrace{T^L(\mathbf{p}\cdots\mathbf{p})}_{\text{orbital part}}\Big]^{s_3},\end{aligned} \tag{154}$$

where the spin part is constructed entirely from spin wave functions defined in the rest frame. It, therefore, carries no orbital information, and the separation between the orbital part and the spin part is explicit. As a result, the LS partial wave amplitudes of the Zemach tensor method are fully consistent with the traditional-LS amplitudes. Likewise, both approaches are non-covariant. To recover Lorentz covariance, covariant tensor methods have been developed, which will be introduced in the next subsection.

2. *Covariant tensor method*

In the covariant tensor method [32, 33], the coupling of angular momenta is described by covariant tensors carrying the $SO(3,1)$ Lorentz indices. The coupling between different angular momentum tensors is implemented by constructing Lorentz structures and contracting the $SO(3,1)$ indices of the participating tensors, thereby producing a new tensor with the desired total angular momentum. To obtain a pure irreducible angular momentum coupling, one typically introduces Lorentz structures transverse to the parent momentum, ensuring that all projections and contractions are performed within the transverse components, and the time component does not participate, avoiding admixtures from other angular momentum components.

Next, the explicit construction of the covariant tensor amplitude is presented. Consider the two-body decay process $3 \to 1+2$, where the spins of the particles are s_3, s_1, and s_2, respectively, and the four-momenta p_3^μ, p_1^μ, and p_2^μ are defined in any frame. The discussion begins with the orbital part that enters the coupling, for which the relevant definitions are introduced first.

For the orbital part, the Zemach tensor method obtains the pure orbital L irreducible object by taking the symmetric traceless part of the relative three-momentum. To make the construction covariant, introduce the relative four-momentum as

$$r^\mu = p_1^\mu - p_2^\mu. \tag{155}$$

A natural starting point is the rank-L symmetric-traceless tensor constructed from r^μ, given by

$$r^L_{\mu_1\cdots\mu_L} \equiv r_{\mu_1}\cdots r_{\mu_L} - \text{trace part}. \tag{156}$$

However, symmetric tracelessness only implies that $r^L_{\mu_1\cdots\mu_L}$ furnishes an $SO(3,1)$ irreducible representation $[\mu_1\cdots\mu_L]$, rather than the pure $SO(3)$ orbital L component. In the COM frame, the time component transforms as an $SO(3)$ scalar, tensor components with one or more time indices behave as scalar factors multiplying lower-rank spatial tensors, and hence contain contributions from orbital angular momenta smaller than L.

To ensure that the constructed tensor indeed lies in the orbital angular momentum L irreducible component, it is required in the COM frame that it has no time components. Equivalently, in any frame, for the symmetric traceless rank-L tensor $r^L_{\mu_1\cdots\mu_L}$ one must subtract a part such that the resulting tensor is orthogonal to the total momentum p_3 in each index, namely it satisfies

$$p_3^{\mu_i}\left[r^L_{\mu_1\cdots\mu_i\cdots\mu_L} - \text{the } p_3\text{-parallel part}\right] = 0, \qquad i = 1,\ldots,L, \tag{157}$$

where the expression in the square brackets is the orbital angular momentum L component in terms of the Lorentz indices $\mu_1\cdots\mu_L$.

The explicit form of the covariant orbital tensor can be obtained by constructing a projection tensor and projecting the relative momentum onto the pure-L component. For this purpose, one first introduces the rank-L integer spin wave function associated with the total momentum p_3, defined as

$$\epsilon^{\mu_1\mu_2\cdots\mu_L}_{\sigma_L}(p_3;L) \equiv \sum_{\sigma_{L-1},\sigma} C^{L-1,\sigma_{L-1};1,\sigma}_{L,\sigma_L}\epsilon^{\mu_1\mu_2\cdots\mu_{L-1}}_{\sigma_{L-1}}(p_3;L-1)\epsilon^{\mu_L}_{\sigma}(p_3), \tag{158}$$

where $C^{L-1,\sigma_{L-1};1,\sigma}_{L,\sigma_L}$ denotes the CGCs and $\epsilon^{\mu_L}_\sigma(p)$ are the polarization four-vectors. The wave function constructed in this way satisfies the Rarita-Schwinger conditions. Namely, it is orthogonal to p_3 in each index and is symmetric and traceless in the Lorentz indices, which can be written as

$$p_{3\mu_1}\epsilon^{\mu_1\mu_2\cdots\mu_L}_{\sigma_L}(p_3;L) = 0, \tag{159}$$

$$\epsilon^{\mu_1\cdots\mu_a\cdots\mu_b\cdots\mu_L}_{\sigma_L}(p_3;L) = \epsilon^{\mu_1\cdots\mu_b\cdots\mu_a\cdots\mu_L}_{\sigma_L}(p_3;L), \tag{160}$$

$$g_{\mu_1\mu_2}\epsilon^{\mu_1\mu_2\cdots\mu_L}_{\sigma_L}(p_3;L) = 0. \tag{161}$$

Accordingly, a projection tensor can be constructed and is defined as [28]

$$P^{\nu_1\cdots\nu_L}_{\mu_1\cdots\mu_L}(p_3;L) \equiv \sum_{\sigma_L}\epsilon^{\nu_1\cdots\nu_L}_{\sigma_L}(p_3;L)\epsilon^{*\sigma_L}_{\mu_1\cdots\mu_L}(p_3;L). \tag{162}$$

This projection tensor also satisfies the Rarita-Schwinger conditions. Acting with this projector tensor on the rank-L tensor formed by the direct product of L relative momenta yields the covariant orbital tensor satisfying the condition in Eq. (157), which is defined as

$$\tilde{t}^{(L)}_{\mu_1\cdots\mu_L}(p_3,r) \equiv P^{\nu_1\cdots\nu_L}_{\mu_1\cdots\mu_L}(p_3;L)r_{\nu_1}\cdots r_{\nu_L}. \tag{163}$$

The tensor $\tilde{t}^{(L)}_{\mu_1\cdots\mu_L}(p_3,r)$ is a rank-L symmetric traceless Lorentz tensor that represents the pure orbital angular momentum L component in the Lorentz indices $\mu_1\cdots\mu_L$. By construction, it is also orthogonal to the total momentum p_3 in each index,

$$p_3^{\mu_1}\tilde{t}^{(L)}_{\mu_1\cdots\mu_L} = 0. \tag{164}$$

Next, several explicit examples are presented. For $L=1$, the projection tensor can be evaluated as

$$\begin{aligned} P^\nu_\mu(p_3;1) &= \sum_\sigma \epsilon^\nu_\sigma(p_3)\epsilon^{*\sigma}_\mu(p_3) \\ &= -g^\nu_{\ \mu} + \frac{p_3^\nu p_{3\mu}}{p_3^2} \\ &= \tilde{g}^\nu_{\ \mu}(p_3). \end{aligned} \tag{165}$$

The result differs from some expressions in the literature due to the placement of upper and lower indices. This is because, in the present convention, the projector tensor is defined using two wave functions with distinct index positions, leading to a notational difference.

Acting with the projector tensor on the relative momentum then yields the covariant orbital tensor for $L=1$, given by

$$\tilde{r}_\mu(p_3) = \tilde{g}^{\nu}_{\mu}(p_3) r_\nu. \tag{166}$$

In the rest frame of particle-3, $k_3^\mu = (m_3, \mathbf{0})$, the condition $k_3^\mu \tilde{r}_\mu(k_3) = 0$ implies $\tilde{r}^0 = 0$, hence $\tilde{r}^\mu = (0, \mathbf{r})$. The projection, therefore, removes the time component, and $\tilde{r}^\mu$ reduces to the relative three-momentum used in the Zemach tensor method. The transverse metric $\tilde{g}^{\nu}_{\mu}(k_3)$ reduces to δ_{ij} in this frame.

For some low orbital angular momenta, the projector can be worked out explicitly; for example,

$$\begin{aligned}
P(p_3;0) &= 1,\\
P^{\nu_1}_{\mu_1}(p_3;1) &= \tilde{g}^{\nu_1}_{\mu_1},\\
P^{\nu_1\nu_2}_{\mu_1\mu_2}(p_3;2) &= \frac{1}{2}\left(\tilde{g}^{\nu_1}_{\mu_1}\tilde{g}^{\nu_2}_{\mu_2} + \tilde{g}^{\nu_1}_{\mu_2}\tilde{g}^{\nu_2}_{\mu_1}\right) - \frac{1}{3}\tilde{g}^{\nu_1\nu_2}\tilde{g}_{\mu_1\mu_2},\\
P^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3}(p_3;3) &= \frac{1}{6}\sum_{P\{\mu_1,\mu_2,\mu_3\}} \tilde{g}^{\nu_1}_{\mu_1}\tilde{g}^{\nu_2}_{\mu_2}\tilde{g}^{\nu_3}_{\mu_3}\\
&\quad - \frac{1}{30}\sum_{P\{\mu_1,\mu_2,\mu_3\}}\Big(\tilde{g}^{\nu_1\nu_2}\tilde{g}_{\mu_1\mu_2}\tilde{g}^{\nu_3}_{\mu_3}\\
&\quad + \tilde{g}^{\nu_2\nu_3}\tilde{g}_{\mu_2\mu_3}\tilde{g}^{\nu_1}_{\mu_1} + \tilde{g}^{\nu_1\nu_3}\tilde{g}_{\mu_1\mu_3}\tilde{g}^{\nu_2}_{\mu_2}\Big),
\end{aligned} \tag{167}$$

where $P\{\mu_1,\mu_2,\mu_3\}$ denotes the set of all 3! permutations of the indices μ_1,μ_2,μ_3, and $\tilde{g}^{\nu}_{\mu}$ is defined in Eq. (165). The corresponding covariant orbital tensor can be computed explicitly as

$$\begin{aligned}
\tilde{t}^{(0)} &= 1,\\
\tilde{t}^{(1)}_{\mu_1} &= \tilde{r}_{\mu_1},\\
\tilde{t}^{(2)}_{\mu_1\mu_2} &= \tilde{r}_{\mu_1}\tilde{r}_{\mu_2} - \frac{1}{3}(\tilde{r}\cdot\tilde{r})\tilde{g}_{\mu_1\mu_2},\\
\tilde{t}^{(3)}_{\mu_1\mu_2\mu_3} &= \tilde{r}_{\mu_1}\tilde{r}_{\mu_2}\tilde{r}_{\mu_3} - \frac{1}{5}(\tilde{r}\cdot\tilde{r})\left(\tilde{g}_{\mu_1\mu_2}\tilde{r}_{\mu_3} + \tilde{g}_{\mu_1\mu_3}\tilde{r}_{\mu_2} + \tilde{g}_{\mu_3\mu_2}\tilde{r}_{\mu_1}\right),
\end{aligned} \tag{168}$$

with

$$\tilde{g}_{\mu\nu} = g_{\mu\mu'}\tilde{g}^{\mu'}_{\nu} = -g_{\mu\nu} + \frac{p_{3\mu}p_{3\nu}}{p_3^2}, \tag{169}$$

where $g_{\mu\nu}$ is the metric tensor, and $\tilde{g}^{\mu}_{\nu}$ and $\tilde{r}_\mu$ are defined in Eqs. (165) and (166). For brevity, the explicit dependence on p_3 is suppressed.

For the spin part, the first step is to specify the external spin wave functions entering the coupling. In the covariant tensor method, the spin wave functions are based on the Rarita-Schwinger construction [83], and canonical spin projection labels are used. For spin-$\frac{1}{2}$, Dirac spinors $u^{\alpha}_{\sigma}(p)$ are used. For spin-1, polarization four-vectors $\epsilon^{\mu}_{\sigma}(p)$ are used. Higher spin wave functions are obtained from $\epsilon^{\mu}_{\sigma}(p)$ and $u^{\alpha}_{\sigma}(p)$ through CGCs couplings as

$$\epsilon^{\mu_1\mu_2\cdots\mu_s}_{\sigma_s}(p;s) = \sum_{\sigma_{s-1},\sigma} C^{s-1,\sigma_{s-1};1,\sigma}_{s,\sigma_s}\epsilon^{\mu_1\mu_2\cdots\mu_{s-1}}_{\sigma_{s-1}}(p;s-1)\epsilon^{\mu_s}_{\sigma}(p), \tag{170}$$

$$u^{\mu_1\mu_2\cdots\mu_s\alpha}_{\sigma_{s+\frac{1}{2}}}(p;s+\tfrac{1}{2}) = \sum_{\sigma_s,\sigma} C^{s,\sigma_s;\frac{1}{2},\sigma}_{s+\frac{1}{2},\sigma_{s+\frac{1}{2}}}\epsilon^{\mu_1\mu_2\cdots\mu_s}_{\sigma_s}(p;s)u^{\alpha}_{\sigma}(p), \tag{171}$$

where $C^{s_1,\sigma_1;s_2,\sigma_2}_{s_3,\sigma_3}$ denotes the CGCs for coupling little group indices. In the first line, the coupling $(s-1)\otimes 1 \to s$ combines a rank-$(s-1)$ integer spin wave function with a spin-1 wave function to produce a rank-s integer spin wave function. Therefore, integer spin wave functions can be generated recursively starting from the spin-1 wave function. In the second line, the coupling $s\otimes\frac{1}{2}\to s+\frac{1}{2}$ combines an integer spin wave function with a Dirac spinor, yielding a half-integer spin wave function.

The basic tensor building blocks for the spin part and the orbital part entering the coupling have now been introduced. In analogy with the Zemach tensor method, the next step is to construct the corresponding coupling structures that implement the angular momentum couplings. The key point is to covariantize the coupling tensors used in the Zemach tensor method and extend them to the $SO(3,1)$ Lorentz indices. Equivalently, this amounts to projecting the product of two angular momentum tensors onto the tensor with the desired total angular momentum.

First, consider the coupling of the two final state spins into a total spin-S, namely $s_1+s_2\to S$. When both spins are integers, the Zemach tensor method performs the coupling by contracting the tensor indices of the two final state spin tensors with the $SO(3)$ invariant tensors δ_{ij} and ε_{ijk}, yielding a total spin-S spin tensor. The resulting rank-S tensor is required to be symmetric and traceless so that it represents a pure spin-S component.

In the $SO(3,1)$ covariant extension, as in the discussion of the orbital part, obtaining a total spin-S tensor is not only a matter of enforcing symmetry and tracelessness. It is also necessary to ensure that, when evaluated in the parent rest frame, the time components vanish so that the tensor corresponds to a pure spin-S component. This can be enforced covariantly by requiring the tensor to be orthogonal to the parent momentum p_3, which can be written as

$$p_3^{\mu_1}T_{\mu_1\mu_2\cdots\mu_S}(S) = 0, \tag{172}$$

where $T_{\mu_1\mu_2\cdots\mu_S}(S)$ denotes the rank-S total spin tensor obtained by coupling the final-state spin wave functions. By incorporating this transversality condition into the coupling structures, the covariant coupling tensors required for integer spin couplings can be specified. They are given by the transverse metric $\tilde{g}^{\mu\nu}(p_3)$ and the transverse antisymmetric tensor $\tilde{\varepsilon}^{\mu\nu\rho}(p_3)$, which are written as

$$\tilde{g}^{\mu\nu}(p_3) = -g^{\mu\nu} + \frac{p_3^\mu p_3^\nu}{p_3^2}, \tag{173}$$

$$\tilde{\varepsilon}^{\mu\nu\rho}(p_3) = \varepsilon^{\mu\nu\rho\kappa} p_{3\kappa}, \tag{174}$$

where indices can be raised and lowered with the metric tensor $g_{\mu\nu}$. Since the present convention distinguishes upper and lower indices explicitly in contractions (with repeated indices summed), the resulting notation may differ from that used in some references. One can readily verify that both tensors are transverse to p_3, and that, in the parent rest frame, their time components vanish so that they are proportional to the invariant tensors δ_{ij} and ε_{ijk}.

As an explicit example, consider the coupling of two spin-1 tensor wave functions, $s_1 = s_2 = 1$, for which

$$1 \otimes 1 = 0 \oplus 1 \oplus 2. \tag{175}$$

Choose the external spin wave functions in the Lorentz four-vector representation to be $\epsilon_\mu^{*\sigma_1}(p_1)$ and $\epsilon_\mu^{*\sigma_2}(p_2)$ for particle-1 and particle-2. By contracting these two external spin wave functions with the projector tensors in the $1\otimes 1 \to S$ channels, one obtains three covariant spin tensors corresponding to the total spins $S = 0, 1, 2$. The $S = 0$ component is

$$\begin{aligned} T^{\sigma_1\sigma_2}(p_3, p_1, p_2; 0) &= \tilde{g}^{\mu\nu}(p_3)\epsilon_\mu^{*\sigma_1}(p_1)\epsilon_\nu^{*\sigma_2}(p_2) \\ &= \Big(-g^{\mu\nu} + \frac{p_3^\mu p_3^\nu}{p_3^2}\Big)\epsilon_\mu^{*\sigma_1}(p_1)\epsilon_\nu^{*\sigma_2}(p_2). \end{aligned} \tag{176}$$

The $S = 1$ component is

$$\begin{aligned} T_\mu^{\sigma_1\sigma_2}(p_3, p_1, p_2; 1) &= g_{\mu\mu'}\tilde{\varepsilon}^{\mu'\nu\rho}(p_3)\epsilon_\nu^{*\sigma_1}(p_1)\epsilon_\rho^{*\sigma_2}(p_2) \\ &= \Big(g_{\mu\mu'}\varepsilon^{\mu'\nu\rho\kappa}p_{3\kappa}\Big)\epsilon_\nu^{*\sigma_1}(p_1)\epsilon_\rho^{*\sigma_2}(p_2). \end{aligned} \tag{177}$$

The $S = 2$ component is

$$\begin{aligned} T_{\mu\nu}^{\sigma_1\sigma_2}(p_3, p_1, p_2; 2) = \bigg(&\frac{1}{2}\Big(\tilde{g}^\rho_{\ \mu}(p_3)\tilde{g}^\kappa_{\ \nu}(p_3) + \tilde{g}^\rho_{\ \nu}(p_3)\tilde{g}^\kappa_{\ \mu}(p_3)\Big) \\ &- \frac{1}{3}\tilde{g}^{\rho\kappa}(p_3)\tilde{g}_{\mu\nu}(p_3)\bigg)\epsilon_\rho^{*\sigma_1}(p_1)\epsilon_\kappa^{*\sigma_2}(p_2). \end{aligned} \tag{178}$$

The coupling structures constructed above are used to couple spins s_1 and s_2 into the total spin-S. Conceptually, they implement the coupling between two angular momentum tensors, and therefore they apply equally well to the next step, namely coupling the total spin-S with the orbital angular momentum L into the parent spin-s_3.

As an example, consider the two-body decay process $3 \to 1+2$, with $s_3 = 1$, $s_1 = 1$, and $s_2 = 1$. Choose the partial wave $(L, S) = (1, 1)$. First, couple the two final-state spins via $1\otimes 1 \to 1$, which yields a total spin-S tensor that can be obtained from Eq. (177). Next, couple the orbital angular momentum $L = 1$ with the total spin $S = 1$ via $1 \otimes 1 \to s_3 = 1$, obtaining a rank-s_3 tensor structure, which can be written as

$$\begin{aligned} &T_{\mu_3}^{\sigma_1\sigma_2}(p_3, p_1, p_2; 1, 1) \\ &= g_{\mu_3\mu_3'}\tilde{\varepsilon}^{\mu_3'\mu\nu}(p_3)T_\mu^{\sigma_1\sigma_2}(p_3, p_1, p_2; 1)\tilde{t}_\nu^{(1)}(p_3, r) \\ &= \Big(g_{\mu_3\mu_3'}\varepsilon^{\mu_3'\mu\nu\kappa}p_{3\kappa}\Big)\Big(g_{\mu\mu'}\varepsilon^{\mu'\mu_1\mu_2\rho}p_{3\rho}\Big)\epsilon_{\mu_1}^{*\sigma_1}(p_1)\epsilon_{\mu_2}^{*\sigma_2}(p_2)\tilde{r}_\nu(p_3) \\ &= -g_{\mu_3\rho}^{\mu_1\mu_2\nu\kappa}p_{3\kappa}p_3^\rho\epsilon_{\mu_1}^{*\sigma_1}(p_1)\epsilon_{\mu_2}^{*\sigma_2}(p_2)\tilde{r}_\nu(p_3), \end{aligned} \tag{179}$$

with

$$\begin{aligned} g_{\mu_3\rho}^{\mu_1\mu_2\nu\kappa} \equiv\ & g_{\mu_3}^{\mu_1}\left(g^{\mu_2\nu}g^\kappa_\rho - g^{\kappa\nu}g^{\mu_2}_\rho\right) - g_{\mu_3}^{\mu_2}\left(g^{\mu_1\nu}g^\kappa_\rho - g^{\kappa\nu}g^{\mu_1}_\rho\right) \\ &+ g^\kappa_{\mu_3}\left(g^{\mu_1\nu}g^{\mu_2}_\rho - g^{\mu_2\nu}g^{\mu_1}_\rho\right). \end{aligned} \tag{180}$$

Finally, contracting this tensor with the external wave function of the initial particle with spin s_3 gives the partial wave amplitude for the decay in the $(L, S) = (1, 1)$ channel, which can be expressed as

$$\begin{aligned} &C_{\sigma_3}^{\sigma_1\sigma_2}(p_3, p_1, p_2; 1, 1) \\ &= \epsilon_{\sigma_3}^{\mu_3}(p_3)T_{\mu_3}^{\sigma_1\sigma_2}(p_3, p_1, p_2; 1, 1) \\ &= -g_{\mu_3\rho}^{\mu_1\mu_2\nu\kappa}p_{3\kappa}p_3^\rho\epsilon_{\sigma_3}^{\mu_3}(p_3)\epsilon_{\mu_1}^{*\sigma_1}(p_1)\epsilon_{\mu_2}^{*\sigma_2}(p_2)\tilde{r}_\nu(p_3). \end{aligned} \tag{181}$$

It follows that the evaluation of a covariant tensor amplitude can be organized into three parts: the external spin wave functions, the orbital part, and the LS coupling part. One can write

$$\begin{aligned} C(p_3, p_1, p_2; L, S) = &\underbrace{P(p_3; s_1, s_2, S)P(p_3; L, S, s_3)}_{LS\ \text{coupling part}} \\ &\times \underbrace{\tilde{t}^{(L)}(p_3, r)}_{\text{orbital part}}\underbrace{u(p_3; s_3)\bar{u}(p_1; s_1)\bar{u}(p_2; s_2)}_{\text{spin wave function part}}. \end{aligned} \tag{182}$$

After obtaining the covariant tensor amplitude, one encounters an immediate issue. In the COM frame, although the LS coupling part, the orbital part, and the parent spin wave function reduce to the same objects as in the Zemach tensor method, the two final-state spin wave functions become $\bar{u}(p_i^*;s_i)$ $(i=1,2)$ and carry the final-state COM momenta, rather than being purely intrinsic-spin objects. The spin wave function part can be written as

$$u(p_3;s_3)\bar{u}(p_1;s_1)\bar{u}(p_2;s_2)\xrightarrow{\text{COM}}u(k_3;s_3)\bar{u}(p_1^*;s_1)\bar{u}(p_2^*;s_2). \tag{183}$$

As a result, in the covariant tensor method, the spin part can carry orbital dependence, and even for fixed (L,S), the resulting partial waves generally differ from the traditional-LS amplitudes defined in the two-body COM frame. Only in the non-relativistic limit, where the final-state spin wave functions reduce to the rest frame spin wave functions, does the covariant tensor method coincide with the Zemach tensor method.

In addition, when half-integer spins are involved, the covariant tensor method typically provides explicit LS coupling structures only for specific processes rather than a general formula applicable to arbitrary spins. To obtain a general computational expression, the covariant projection tensor method has been developed [48]. This method constructs covariant projection tensors for angular momentum, which enables a systematic evaluation of the required coupling structures for arbitrary spins. The next subsection presents the general method for computing the LS coupling part.

B. Covariant projection tensor method

In this subsection, the covariant projection tensor method is introduced [48]. In this method, the $SO(3,1)$ coupling relations are used to construct the covariant projection tensor. This covariant projection tensor plays the role of angular momentum coupling in the $SO(3,1)$ Lorentz representation. In the covariant projection tensor method, the coupling in the $SO(3,1)$ representation is implemented through this projection tensor. In practice, two schemes are used depending on the COM definition, referred to as the GS scheme and the PS scheme. The construction of these two schemes is discussed next.

1. *Covariant projection tensor method in the GS scheme*

Both the GS scheme and the PS scheme require the covariant projection tensors and the covariant orbital tensors. Therefore, the discussion begins with the definitions of these two covariant tensors.

In the covariant projection tensor method, angular momentum coupling is required to be implemented directly in the Lorentz representation of $SO(3,1)$. This requires that the little group coupling be transferred to Lorentz indices. The following discussion explains how angular momentum coupling is realized in the Lorentz representation and how it is related to the little group coupling. A bridge between the little group representation and the Lorentz representation is established by single-particle states, field operators, and spin wave functions. The single-particle state can be defined using the creation operator as

$$|\mathbf{p},s\sigma\rangle\equiv a^\dagger(\mathbf{p},s\sigma)|0\rangle, \tag{184}$$

where $|0\rangle$ is the vacuum state. By Eq. (31), the states transform under a Lorentz transformation Λ as follows,

$$U[\Lambda]|\mathbf{p},s\sigma\rangle=\sum_{\sigma'}D^{\sigma'(s)}_{\sigma}(W(\Lambda,p))|\mathbf{p}',s\sigma'\rangle,\quad p'=\Lambda p, \tag{185}$$

where $W(\Lambda,p)$ is the little group rotation, and $D^{\sigma'(s)}_{\sigma}(W(\Lambda,p))$ is the rotation matrix in the spin-s representation. For a quantum field operator in the representation $[\ell]$, it can be written as

$$\Phi^\ell(p)=\sum_\sigma u^\ell_\sigma(p;s)a(\mathbf{p},s\sigma), \tag{186}$$

where $a(\mathbf{p},s\sigma)$ is the annihilation operator and $u^\ell_\sigma(p;s)$ is the spin wave function in the corresponding representation $[\ell]$. Under a Lorentz transformation Λ, the field operator $\Phi^\ell(p)$ transforms as follows,

$$U[\Lambda]\Phi^\ell(p)U^{-1}[\Lambda]=D^\ell_{\ell'}(\Lambda^{-1})\Phi^{\ell'}(\Lambda p), \tag{187}$$

where $D^\ell_{\ell'}(\Lambda^{-1})$ is the Lorentz transformation matrix corresponding to the representation $[\ell]$. The spin wave function $u^\ell_\sigma(p;s)$ is defined as the matrix element of the field operator between the vacuum and the single-particle state

$$u^\ell_\sigma(p;s)\equiv\langle 0|\Phi^\ell(p)|\mathbf{p},s\sigma\rangle, \tag{188}$$

where the spin wave function must satisfy the normalization condition

$$\bar{u}^{\sigma_1}_\ell(p;s)u^\ell_{\sigma_2}(p;s)=\delta^{\sigma_1}_{\sigma_2}, \tag{189}$$

where $\bar{u}^\sigma_\ell(p;s)$ is defined as

$$\bar{u}^\sigma_\ell(p;s)\equiv\langle\mathbf{p},s\sigma|\bar{\Phi}_\ell(p)|0\rangle, \tag{190}$$

with

$$\bar{\Phi}_\ell(p) \equiv \sum_\sigma \bar{u}_\ell^\sigma(p;s)a^\dagger(\mathbf{p},s\sigma). \tag{191}$$

Using Eqs. (185) and (187), one can show that under a Lorentz transformation Λ,

$$\sum_{\sigma'} D_\sigma^{\sigma'(s)}(W(\Lambda,p))u_{\sigma'}^\ell(\Lambda p;s) = D_{\ell'}^\ell(\Lambda)u_\sigma^{\ell'}(p;s), \tag{192}$$

$$\sum_{\sigma'} D_\sigma^{\sigma'(s)*}(W(\Lambda,p))\bar{u}_\ell^{\sigma'}(\Lambda p;s) = D_\ell^{\ell'}(\Lambda^{-1})\bar{u}_{\ell'}^\sigma(p;s). \tag{193}$$

This relation contains both the induced little group rotation and the $SO(3,1)$ representation matrix, and it provides an explicit correspondence between little group indices and Lorentz indices. With this correspondence, the angular momentum coupling formulated in the little group $SO(3)$ representation can be rewritten as a tensor coupling in the Lorentz $SO(3,1)$ representation. This allows the covariant angular momentum projection and contraction to be implemented directly at the level of Lorentz indices.

The two-body decay process $3 \to 1+2$ is considered, where all particles are massive with spins s_3, s_1, and s_2. In an arbitrary frame, their momenta are p_3, p_1, and p_2. The COM frame is defined as the rest frame of particle-3. The canonical-standard boost $L_{3,c}(\mathbf{p}_3)$ is chosen such that

$$p_3 = L_{3,c}(\mathbf{p}_3)k_3, \qquad k_3 = (m_3,0,0,0), \tag{194}$$

where m_3 is the mass of particle-3, and the subscript 3 on the canonical-standard boost indicates that it is the standard boost associated with particle-3. By applying the inverse boost, the momenta in the COM frame are obtained as

$$k_3 = L_{3,c}^{-1}(\mathbf{p}_3)p_3, \qquad p_1^* = L_{3,c}^{-1}(\mathbf{p}_3)p_1, \qquad p_2^* = L_{3,c}^{-1}(\mathbf{p}_3)p_2. \tag{195}$$

Starting with the coupling of spin-s_1 and spin-s_2 to form the total spin-S. In the COM frame, only intrinsic spin coupling is considered, and thus the orbital part is omitted. The spin labels are defined with respect to the little group associated with the total momentum in this frame, which is fixed by k_3. Accordingly, Eq. (42) expresses the spin coupling as

$$|S\sigma_S\rangle = \sum_{\sigma_1\sigma_2} C_{S,\sigma_S}^{s_1,\sigma_1,s_2,\sigma_2}|s_1\sigma_1\rangle|s_2\sigma_2\rangle, \tag{196}$$

where the particle states implicitly refer to the standard momentum states with $\mathbf{k}=0$, denoted by $|\mathbf{k},s\sigma\rangle$. For brevity, the corresponding standard momentum labels are suppressed. This explicitly indicates that the two spins are coupled within the little group representation through the CGCs.

To implement the spin coupling at the level of Lorentz representation, the coupling structure needs to be defined in the COM frame with respect to the parent standard momentum k_3. This is because, at the level of particle states, the spin coupling occurs in the little group associated with the parent standard momentum k_3. Consequently, the translation from the corresponding little group coupling to the Lorentz representation coupling should be performed at the same standard momentum. For this purpose, introduce the field operators at the standard momentum k_3 corresponding to spin-s_1 and spin-s_2, transforming in the Lorentz representations $[\ell_1]$ and $[\ell_2]$, written as

$$\Phi^{\ell_1}(k_3) = \sum_{\sigma_1} u_{\sigma_1}^{\ell_1}(k_3;s_1)a(\mathbf{k},s_1\sigma_1), \tag{197}$$

$$\Phi^{\ell_2}(k_3) = \sum_{\sigma_2} u_{\sigma_2}^{\ell_2}(k_3;s_2)a(\mathbf{k},s_2\sigma_2). \tag{198}$$

It should be emphasized that the spin wave functions in these expressions are not the external spin wave functions, but rather the spin wave functions at the standard momentum k_3 used to construct the coupling structures. They can be defined as

$$u_{\sigma_1}^{\ell_1}(k_3;s_1) \equiv \langle 0|\Phi^{\ell_1}(k_3)|s_1\sigma_1\rangle, \quad u_{\sigma_2}^{\ell_2}(k_3;s_2) \equiv \langle 0|\Phi^{\ell_2}(k_3)|s_2\sigma_2\rangle. \tag{199}$$

For the total spin-S obtained after coupling, the corresponding Lorentz representation is denoted by $[\ell_S]$. The field operator at the standard momentum k_3 is taken as

$$\bar{\Phi}^{\ell_S}(k_3) = \sum_{\sigma_S} \bar{u}_{\sigma_S}^{\ell_S}(k_3;S)a^\dagger(\mathbf{k},S\sigma_S). \tag{200}$$

The corresponding spin wave functions are defined by:

$$\bar{u}_{\ell_S}^{\sigma_S}(k_3;S) \equiv \langle S\sigma_S|\bar{\Phi}_{\ell_S}(k_3)|0\rangle. \tag{201}$$

On this basis, the Lorentz representations $[\ell_1]$ and $[\ell_2]$ carried by the spin wave functions are coupled into the Lorentz representation $[\ell_S]$ associated with the total spin S, which can be written as

$$[\ell_1]\otimes[\ell_2] \to [\ell_S], \qquad |s_1-s_2| \le S \le s_1+s_2. \tag{202}$$

Using Eqs. (196), (199), and (201), the covariant projection tensor is then introduced as

$$\begin{aligned}&P^{\ell_1\ell_2}_{\ell_S}(k_3;S,s_1,s_2)\\&=\langle 0|\bar{\Phi}_{\ell_S}(k_3)\Phi^{\ell_1}(k_3)\Phi^{\ell_2}(k_3)|0\rangle\\&=\langle S\sigma_S|\bar{u}^{\sigma_S}_{\ell_S}(k_3;S)u^{\ell_1}_{\sigma_1}(k_3;s_1)u^{\ell_2}_{\sigma_2}(k_3;s_2)|s_1\sigma_1\rangle|s_2\sigma_2\rangle\\&=\sum_{\sigma_1\sigma_2\sigma_S}C^{s_1,\sigma_1;s_2,\sigma_2}_{S,\sigma_S}\bar{u}^{\sigma_S}_{\ell_S}(k_3;S)u^{\ell_1}_{\sigma_1}(k_3;s_1)u^{\ell_2}_{\sigma_2}(k_3;s_2).\end{aligned}\tag{203}$$

The meaning of this covariant projection tensor is that it rewrites, at the standard momentum k_3 in the COM frame, the spin coupling implemented by CGCs using little group indices into an equivalent covariant object that involves only Lorentz indices. That is, the spin coupling is no longer written explicitly as a contraction over little group indices. Instead, it is encoded into contractions over Lorentz indices, thereby translating the coupling in the little group representation into the coupling in the Lorentz representation.

Its construction can be understood in the following three steps:

1. The rest frame spin wave functions at the standard momentum k_3, namely $u^{\ell_1}_{\sigma_1}(k_3;s_1)$ and $u^{\ell_2}_{\sigma_2}(k_3;s_2)$, provide the link between Lorentz representations and the corresponding little group representations. In this way, the Lorentz representations associated with spin-s_1 and spin-s_2 are mapped into the same little group representation fixed by k_3.

2. Within this common little group representation, the two spins are coupled by the CGCs, yielding the little group coupling result with total spin S.

3. The rest frame spin wave function for total spin S, namely $\bar{u}^{\sigma_S}_{\ell_S}(k_3;S)$, maps the coupled little group representation back to the Lorentz representation associated with the total spin S.

Therefore, $P^{\ell_1\ell_2}_{\ell_S}$ can be viewed as a covariant projection tensor that projects the tensor product of two Lorentz representations onto the Lorentz representation corresponding to total spin-S. When it is completely contracted with the external wave functions over Lorentz indices, it is equivalent to the action of the CGCs in the little group representation.

Under a Lorentz transformation Λ, the projector tensor in Eq. (203) satisfies

$$\begin{aligned}&P^{\ell_1\ell_2}_{\ell_S}(\Lambda k_3;S,s_1,s_2)\\&=D^{\ell_1}_{\ell_1'}(\Lambda)D^{\ell_2}_{\ell_2'}(\Lambda)P^{\ell_1'\ell_2'}_{\ell_S'}(k_3;S,s_1,s_2)D^{\ell_S'}_{\ell_S}(\Lambda^{-1})\\&=\sum_{\sigma_1\sigma_2\sigma_S}C^{s_1,\sigma_1;s_2,\sigma_2}_{S,\sigma_S}\bar{u}^{\sigma_S}_{\ell_S}(\Lambda k_3;S)u^{\ell_1}_{\sigma_1}(\Lambda k_3;s_1)u^{\ell_2}_{\sigma_2}(\Lambda k_3;s_2),\end{aligned}\tag{204}$$

which demonstrates the covariance of the projection tensor. Consequently, when transitioning back to any frame, it retains the same form and can be expressed as

$$\begin{aligned}&P^{\ell_1\ell_2}_{\ell_S}(p_3;S,s_1,s_2)\\&=\sum_{\sigma_1\sigma_2\sigma_S}C^{s_1,\sigma_1;s_2,\sigma_2}_{S,\sigma_S}\bar{u}^{\sigma_S}_{\ell_S}(p_3;S)u^{\ell_1}_{\sigma_1}(p_3;s_1)u^{\ell_2}_{\sigma_2}(p_3;s_2).\end{aligned}\tag{205}$$

Once a specific Lorentz realization is chosen for the representation $[\ell]$, the corresponding covariant projection tensor can be explicitly evaluated. For example, by selecting the Lorentz four-vector representation with indices $\mu,\nu,\cdots$, in the case where $s_1=1,\ s_2=1$, and $S=1$, the covariant projection tensor can be expressed as

$$\begin{aligned}&P^{\mu_1\mu_2}_{\mu_S}(p_3;1,1,1)\\&=\sum_{\sigma_1\sigma_2\sigma_S}C^{1,\sigma_1;1,\sigma_2}_{1,\sigma_S}\bar{u}^{\sigma_S}_{\mu_S}(p_3;1)u^{\mu_1}_{\sigma_1}(p_3;1)u^{\mu_2}_{\sigma_2}(p_3;1)\\&=\sum_{\sigma_1\sigma_2\sigma_S}C^{1,\sigma_1;1,\sigma_2}_{1,\sigma_S}\epsilon^{*\sigma_S}_{\mu_S}(p_3)\epsilon^{\mu_1}_{\sigma_1}(p_3)\epsilon^{\mu_2}_{\sigma_2}(p_3)\\&=\frac{\mathrm{i}}{\sqrt{2}m_3}g_{\mu_S\mu_S'}\varepsilon^{\mu_S'\mu_1\mu_2\rho}p_{3\rho}.\end{aligned}\tag{206}$$

One can see that after factoring out the overall normalization constant, this covariant projection tensor is precisely the $1\otimes 1\to 1$ coupling structure used in the covariant tensor method in Eq. (177). Therefore, by constructing such projection tensors, the coupling structures required in the covariant tensor method can be obtained systematically for arbitrary spins.

Besides projecting spin-s_1 and spin-s_2 onto the total spin-S, this covariant projection tensor has a more general meaning. It defines an $SO(3,1)$ covariant tensor that maps the direct product of two angular momentum representations $[\ell_{j_1}]$ and $[\ell_{j_2}]$ into the representation $[\ell_J]$. Therefore, it not only serves as the covariant projection tensor for coupling s_1 and s_2 to S, but it can also be used to project the Lorentz representation $[\ell_L]$ of the orbital part L together with the Lorentz representation $[\ell_S]$ of the total spin part S onto the Lorentz representation $[\ell_3]$ associated with the parent spin-s_3. The corresponding covariant projection tensor can be written as

$$\begin{aligned}&P^{\ell_L\ell_S}_{\ell_3}(p_3;s_3,L,S)\\&=\sum_{\sigma_S\sigma_L\sigma_3}C^{L,\sigma_L;S,\sigma_S}_{s_3,\sigma_3}\bar{u}^{\sigma_3}_{\ell_3}(p_3;s_3)u^{\ell_L}_{\sigma_L}(p_3;L)u^{\ell_S}_{\sigma_S}(p_3;S).\end{aligned}\tag{207}$$

Since the representation $[\ell_S]$ of the total spin-S obtained from the projection has been given above, the next step is to discuss the orbital part in the Lorentz representation $[\ell_L]$.

In the two-body COM frame, the orbital part is completely determined by the relative momentum of the final-state particles. Therefore, only the momentum dependence of the particle-1 and particle-2 states is retained, while the spin labels are suppressed. From Eq. (44), the orbital angular momentum state, defined in the little group representation associated with the total momentum k_3, can be introduced through spherical harmonics as

$$|L\sigma_L\rangle = \int d\Omega Y^L_{\sigma_L}(\Omega)|\mathbf{p}_1^*\mathbf{p}_2^*\rangle, \tag{208}$$

where $|\mathbf{p}_1^*\mathbf{p}_2^*\rangle$ denotes the two-particle state without spin labels, specifically $|\mathbf{p}_1^*,0\rangle$ and $|\mathbf{p}_2^*,0\rangle$, and the spin labels of particle-1 and particle-2 are omitted. The orbital angular momentum state is denoted as $|\mathbf{k},L\sigma_L\rangle$, and for the sake of brevity, the standard momentum label of this orbital state is also omitted. The spherical harmonic can be viewed as the $SO(3)$ representation component of the rank-L symmetric traceless tensor built from the relative three-momentum $\mathbf{p}_1^*-\mathbf{p}_2^*$. Schematically, this structure is represented by

$$Y^L_{\sigma_L}(\Omega) \simeq (\mathbf{p}_1^*-\mathbf{p}_2^*)^L_{i_1\cdots i_L} \equiv (\mathbf{p}_1^*-\mathbf{p}_2^*)_{i_1}\cdots(\mathbf{p}_1^*-\mathbf{p}_2^*)_{i_L} - \text{trace part}, \tag{209}$$

where $i_1,\ldots,i_L$ are spatial indices and $(\mathbf{p}_1^*-\mathbf{p}_2^*)^L_{i_1\cdots i_L}$ denotes the **symmetric traceless** rank-L tensor constructed from the relative three-momentum. To obtain the covariant formulation, the spherical harmonic in Eq. (44) is replaced by the rank-L tensor built from the relative four-momentum $(p_1^*-p_2^*)_\mu$. For this purpose, in the COM frame with total momentum k_3, a Poincaré particle state is introduced, whose components are characterized by Lorentz indices $\mu_1\cdots\mu_L$. It is defined as

$$|\mu_1\cdots\mu_L\rangle \equiv \int d\Omega (p_1^*-p_2^*)^L_{\mu_1\cdots\mu_L}|\mathbf{p}_1^*\mathbf{p}_2^*\rangle, \tag{210}$$

where $(p_1^*-p_2^*)^L_{\mu_1\cdots\mu_L}$ denotes the **symmetric traceless** rank-L tensor constructed from the relative four-momentum, and, for brevity, the standard momentum label of the orbital state $|\mathbf{k},\mu_1\cdots\mu_L\rangle$ is suppressed.

By construction, $(p_1^*-p_2^*)^L_{\mu_1\cdots\mu_L}$ is a rank-L symmetric traceless tensor, which is an $SO(3,1)$ irreducible representation $[\mu_1\cdots\mu_L]$. Since the time component transforms as an $SO(3)$ scalar in the COM frame, tensor components with one or more time indices behave as scalar factors multiplying lower-rank spatial tensors and hence contain contributions from orbital angular momenta smaller than L. Consequently, the restriction of the representation $[\mu_1\cdots\mu_L]$ to the $SO(3)$ little group of the total momentum k_3 is reducible and decomposes into a direct sum of the $SO(3)$ irreducible components $[\ell_i]$ as

$$[\mu_1\cdots\mu_L] = \bigoplus_i [\ell_i], \tag{211}$$

where i labels all possible orbital components. The corresponding pure L covariant orbital angular momentum state is denoted by $|\ell_L\rangle$, which represents the orbital angular momentum L irreducible component of the $SO(3)$ little group. It is $SO(3)$-equivalent to the orbital angular momentum state $|L\sigma_L\rangle$ in Eq. (208) by

$$|\ell_L\rangle \simeq |L\sigma_L\rangle, \tag{212}$$

where both objects describe the orbital angular momentum L irreducible component, but in different realizations. The state $|\ell_L\rangle$ is written in the covariant Lorentz index realization, whereas $|L\sigma_L\rangle$ is written in the little group index realization.

Next, the corresponding projection tensor is introduced, which projects the Lorentz representation $[\mu_1\cdots\mu_L]$ onto the Lorentz representation $[\ell_L]$ that corresponds to the pure orbital angular momentum L irreducible component. From Eqs. (210) and (211), the state $|\mu_1\cdots\mu_L\rangle$ is known to contain the orbital angular momentum L component. It can be defined as

$$|\mu_1\cdots\mu_L\rangle \equiv \bar{\Phi}_{\mu_1\cdots\mu_L}(k_3)|0\rangle, \tag{213}$$

where $\bar{\Phi}_{\mu_1\cdots\mu_L}(k_3)$ is the field operator that creates the state in Eq. (210) from the vacuum, carrying Lorentz indices $\mu_1\cdots\mu_L$ and total momentum k_3. Similarly, the pure orbital angular momentum L state $|\ell_L\rangle$ can be defined as

$$|\ell_L\rangle \equiv \bar{\Phi}_{\ell_L}(k_3)|0\rangle, \tag{214}$$

where $\bar{\Phi}_{\ell_L}(k_3)$ is the field operator that creates the state in the Lorentz representation $[\ell_L]$ from the vacuum, carrying Lorentz indices ℓ_L and total momentum k_3. Therefore, a projection operator exists that projects the Lorentz representation $[\mu_1\cdots\mu_L]$ onto the pure orbital angular momentum L Lorentz representation $[\ell_L]$. The corresponding orbital projection operator is defined as

$$\begin{aligned} P^{\mu_1\cdots\mu_L}_{\ell_L}(k_3;L) &\equiv \langle\mu_1\cdots\mu_L|\ell_L\rangle \\ &= \langle 0|\bar{\Phi}_{\ell_L}(k_3)\Phi^{\mu_1\cdots\mu_L}(k_3)|0\rangle. \end{aligned} \tag{215}$$

Using the covariant projection tensor in Eq. (203), this projection can be implemented recursively in a fixed coupling order. For $L=0$, there is no Lorentz index. In the Lorentz representation associated with orbital angular momentum $L=0$, the corresponding projection tensor is a scalar.

For $L=1$, the corresponding projection tensor is equivalent to coupling the spin-1 Lorentz representation $[\mu_1]$ with that of spin-0 to obtain the Lorentz representation $[\ell_1]$. From Eq. (203), the covariant projection tensor can be written as

$$\begin{aligned} P^{\mu_1}_{\ell_1}(k_3;1,1,0) &= \sum_{\sigma_1\sigma} C^{1,\sigma_1;0}_{1,\sigma}\bar{u}^{\sigma}_{\ell_1}(k_3;1)u^{\mu_1}_{\sigma_1}(k_3;1) \\ &= \sum_{\sigma_1}\bar{u}^{\sigma_1}_{\ell_1}(k_3;1)u^{\mu_1}_{\sigma_1}(k_3;1). \end{aligned} \tag{216}$$

For an arbitrary $L>1$, the last $L-1$ indices $\mu_2\cdots\mu_L$ are first projected onto an intermediate orbital angular momentum $L-1$ Lorentz representation $[\ell_{L-1}]$, and the remaining index μ_1 is then treated as a spin-1 Lorentz representation $[\mu_1]$. It is coupled with $[\ell_{L-1}]$ through the projection tensor that realizes $1\otimes(L-1)\to L$. This yields

$$P^{\mu_1\cdots\mu_L}_{\ell_L}(k_3;L) = P^{\mu_1\ell_{L-1}}_{\ell_L}(k_3;L,1,L-1)P^{\mu_2\cdots\mu_L}_{\ell_{L-1}}(k_3;L-1), \tag{217}$$

where $P^{\mu_1\ell_{L-1}}_{\ell_L}(k_3;L,1,L-1)$ is the covariant projection tensor implementing $1\otimes(L-1)\to L$. By Eq. (203), it can be written as

$$\begin{aligned} &P^{\mu_1\ell_{L-1}}_{\ell_L}(k_3;L,1,L-1) \\ &= \sum_{\sigma_1\sigma_{L-1}\sigma_L} C^{1,\sigma_1;L-1,\sigma_{L-1}}_{L,\sigma_L}\bar{u}^{\sigma_L}_{\ell_L}(k_3;L)u^{\mu_1}_{\sigma_1}(k_3;1)u^{\ell_{L-1}}_{\sigma_{L-1}}(k_3;L-1). \end{aligned} \tag{218}$$

Therefore, $P^{\mu_1\cdots\mu_L}_{\ell_L}(k_3;L)$ is obtained through successive couplings implemented by the projection tensor. This projection tensor can then be used to project the rank-L tensor, constructed from the relative four-momentum with indices $\mu_1\cdots\mu_L$, onto the desired orbital-L Lorentz representation $[\ell_L]$. Consequently, the covariant orbital tensor is defined as

$$t^L_{\ell_L}(k_3,p_1^*-p_2^*) \equiv P^{\mu_1\cdots\mu_L}_{\ell_L}(k_3;L)(p_1^*-p_2^*)_{\mu_1}\cdots(p_1^*-p_2^*)_{\mu_L}, \tag{219}$$

where $t^L_{\ell_L}(k_3,p_1^*-p_2^*)$ is manifestly covariant. In any frame, it keeps the same form and can be written as

$$t^L_{\ell_L}(p_3,p_1-p_2) \equiv P^{\mu_1\cdots\mu_L}_{\ell_L}(p_3;L)(p_1-p_2)_{\mu_1}\cdots(p_1-p_2)_{\mu_L}. \tag{220}$$

After fixing a specific Lorentz realization $[\ell_L]$ of the pure orbital angular momentum L, the corresponding covariant orbital tensors can be evaluated explicitly. As a representative example, take $[\ell_L]$ to be realized by Lorentz four-vector indices, namely by Lorentz tensor indices $\mu_1\cdots\mu_L$ that carry the pure L component. In this case, after simplification, the associated projection tensor can be written as

$$P^{\nu_1\cdots\nu_L}_{\mu_1\cdots\mu_L}(p_3;L) = \sum_{\sigma_L}\epsilon^{\nu_1\cdots\nu_L}_{\sigma_L}(p_3;L)\epsilon^{*\sigma_L}_{\mu_1\cdots\mu_L}(p_3;L), \tag{221}$$

where $\epsilon^{\nu_1\cdots\nu_L}_{\sigma_L}(p_3;L)$ is the spin wave function satisfying the Rarita–Schwinger conditions, given in Eq. (158). One can see that this projection tensor is exactly the orbital projector used for the orbital part in Eq. (162) of the covariant tensor method. It projects the rank-L tensor built from the relative momentum onto the subspace that is symmetric, traceless, and transverse to the parent momentum in each index, thereby extracting the irreducible component corresponding to the pure orbital angular momentum L.

With this choice, the covariant orbital tensors for several low-L cases can be written as

$$\begin{aligned} t^0 &= 1, \\ t^1_{\mu_1} &= \Big(-g^{\nu_1}_{\mu_1}+\frac{p_3^{\nu_1}p_{3\mu_1}}{p_3^2}\Big)(p_1-p_2)_{\nu_1}, \\ t^2_{\mu_1\mu_2} &= \Big(-g^{\nu_1}_{\mu_1}+\frac{p_3^{\nu_1}p_{3\mu_1}}{p_3^2}\Big)\Big(-g^{\nu_2}_{\mu_2}+\frac{p_3^{\nu_2}p_{3\mu_2}}{p_3^2}\Big) \\ &\quad\times(p_1-p_2)_{\nu_1}(p_1-p_2)_{\nu_2} \\ &\quad-\frac{1}{3}\Big[\Big(-g^{\nu_1\nu_2}+\frac{p_3^{\nu_1}p_3^{\nu_2}}{p_3^2}\Big)(p_1-p_2)_{\nu_1}(p_1-p_2)_{\nu_2}\Big] \\ &\quad\times\Big(-g_{\mu_1\mu_2}+\frac{p_{3\mu_1}p_{3\mu_2}}{p_3^2}\Big). \end{aligned} \tag{222}$$

It follows that the covariant orbital tensors obtained in this realization coincide exactly with the orbital tensors in Eq. (168) of the covariant tensor method. Equivalently, this projection tensor projects the rank-L tensor constructed from the relative momentum onto the Lorentz representation $[\ell_L]$ associated with the corresponding pure L irreducible component.

After defining the covariant tensors in Eqs. (204), (207), and (219), the construction of the corresponding partial wave amplitudes can be formulated. First, recall the partial wave amplitude in the covariant tensor method in Eq. (182). It can be written as a product of three pieces: the LS coupling part, the orbital part, and the external spin wave function part. The LS coupling part and the orbital part are precisely given by the total spin-S projection tensor, the orbital-spin coupling projection tensor, and the covariant orbital tensor defined above. Therefore, by combining these building blocks, the covariant LS coupling structure is defined as

$$\Gamma^{\ell_1\ell_2}_{\ell_3}(k_3,p_1^*,p_2^*;L,S)$$
$$\equiv P^{\ell_L\ell_S}_{\ell_3}(k_3;s_3,L,S)P^{\ell_1\ell_2}_{\ell_S}(k_3;S,s_1,s_2)t^L_{\ell_L}(k_3;p_1^*-p_2^*). \quad (223)$$

By contracting this covariant LS coupling structure with the external spin wave functions over the Lorentz indices, the LS partial wave amplitude in the GS scheme is obtained. It can be written as

$$\begin{aligned}&C^{\sigma_1\sigma_2}_{\sigma_3}(k_3,p_1^*,p_2^*;L,S)\\ &\equiv \Gamma^{\ell_1\ell_2}_{\ell_3}(k_3,p_1^*,p_2^*;L,S)\times u^{\ell_3}_{\sigma_3}(k_3;s_3)\bar{u}^{\sigma_1}_{\ell_1}(p_1^*;s_1)\bar{u}^{\sigma_2}_{\ell_2}(p_2^*;s_2)\\ &= P^{\ell_1\ell_2}_{\ell_S}(k_3;S,s_1,s_2)\bar{u}^{\sigma_1}_{\ell_1}(p_1^*;s_1)\bar{u}^{\sigma_2}_{\ell_2}(p_2^*;s_2)\\ &\quad\times P^{\ell_L\ell_S}_{\ell_3}(k_3;s_3,L,S)t^L_{\ell_L}(k_3;p_1^*-p_2^*)\\ &\quad\times u^{\ell_3}_{\sigma_3}(k_3;s_3).\end{aligned} \quad (224)$$

The physical content of this construction can be organized into three steps:

1. $P^{\ell_1\ell_2}_{\ell_S}(k_3;S,s_1,s_2)\bar{u}^{\sigma_1}_{\ell_1}(p_1^*;s_1)\bar{u}^{\sigma_2}_{\ell_2}(p_2^*;s_2)$ represents the projection of the representation $[\ell_1]\otimes[\ell_2]$ carried by the two final state external spin wave functions onto the total spin-S representation $[\ell_S]$, thereby implementing the spin coupling $s_1\otimes s_2\to S$.

2. $P^{\ell_L\ell_S}_{\ell_3}(k_3;s_3,L,S)t^L_{\ell_L}(k_3;p_1^*-p_2^*)$ represents further projecting $[\ell_L]\otimes[\ell_S]$ onto the representation $[\ell_3]$ of the parent spin s_3, implementing the coupling $L\otimes S\to s_3$.

3. Project the coupling result obtained above onto the parent external spin component σ_3 to finally obtain a scalar partial wave amplitude with external spin labels $(\sigma_1,\sigma_2,\sigma_3)$.

Since the entire expression implements the coupling and projection explicitly through Lorentz index contractions, it is covariant in form. In any frame, it retains the same form and can be written as

$$\begin{aligned}C^{\sigma_1\sigma_2}_{\sigma_3}(p_3,p_1,p_2;L,S) = &\underbrace{\Gamma^{\ell_1\ell_2}_{\ell_3}(p_3,p_1,p_2;L,S)}_{\text{covariant } LS \text{ coupling structure}}\\ &\times\underbrace{u^{\ell_3}_{\sigma_3}(p_3;s_3)\bar{u}^{\sigma_1}_{\ell_1}(p_1;s_1)\bar{u}^{\sigma_2}_{\ell_2}(p_2;s_2)}_{\text{general-spin part}}.\end{aligned} \quad (225)$$

The GS scheme is, in essence, fully equivalent to the covariant tensor method. When the Lorentz representations $[\ell]$ are chosen to be the corresponding four-vector realizations, this equivalence becomes explicit.

For example, consider the two-body decay with $s_3=1$, $s_1=1$, and $s_2=1$, and select the partial wave $(L,S)=(1,1)$. In this case, the total spin-S covariant projection tensor can be written as

$$\begin{aligned}P^{\mu_1\mu_2}_{\mu_S}(p_3;1,1,1) &= \sum_{\sigma_1\sigma_2\sigma_S} C^{1,\sigma_1;1,\sigma_2}_{1,\sigma_S}\epsilon^{*\sigma_S}_{\mu_S}(p_3)\epsilon^{\mu_1}_{\sigma_1}(p_3)\epsilon^{\mu_2}_{\sigma_2}(p_3)\\ &= \frac{\mathrm{i}}{\sqrt{2}m_3}g_{\mu_S\mu_S'}\varepsilon^{\mu_S'\mu_1\mu_2\rho}p_{3\rho}.\end{aligned} \quad (226)$$

The orbital-spin covariant projection tensor can be written as

$$\begin{aligned}P^{\mu_L\mu_S}_{\mu_3}(p_3;1,1,1) &= \sum_{\sigma_L\sigma_S\sigma_3} C^{1,\sigma_L;1,\sigma_S}_{1,\sigma_3}\epsilon^{*\sigma_3}_{\mu_3}(p_3)\epsilon^{\mu_L}_{\sigma_L}(p_3)\epsilon^{\mu_S}_{\sigma_S}(p_3)\\ &= \frac{\mathrm{i}}{\sqrt{2}m_3}g_{\mu_3\mu_3'}\varepsilon^{\mu_3'\mu_L\mu_S\rho}p_{3\rho},\end{aligned} \quad (227)$$

and the covariant orbital tensor can be expressed as

$$t^1_{\mu_L} = \Big(-g^{\nu_L}_{\mu_L}+\frac{p_3^{\nu_L}p_{3\mu_L}}{p_3^2}\Big)(p_1-p_2)_{\nu_L}. \quad (228)$$

After contracting the covariant LS coupling structure with the external spin wave functions, the final partial wave amplitude can be written as

$$\begin{aligned}&C^{\sigma_1\sigma_2}_{\sigma_3}(p_3,p_1,p_2;1,1)\\ &= \Gamma^{\mu_1\mu_2}_{\mu_3}(p_3,p_1,p_2;1,1)\times\epsilon^{\mu_3}_{\sigma_3}(p_3)\epsilon^{*\sigma_1}_{\mu_1}(p_1)\epsilon^{*\sigma_2}_{\mu_2}(p_2)\\ &\propto g^{\mu_1\mu_2\mu_L\kappa}_{\mu_3\rho}p_{3\kappa}p_3^{\rho}\Big(-g^{\nu_L}_{\mu_L}+\frac{p_3^{\nu_L}p_{3\mu_L}}{p_3^2}\Big)(p_1-p_2)_{\nu_L}\\ &\quad\times\epsilon^{\mu_3}_{\sigma_3}(p_3)\epsilon^{*\sigma_1}_{\mu_1}(p_1)\epsilon^{*\sigma_2}_{\mu_2}(p_2),\end{aligned} \quad (229)$$

with

$$\begin{aligned}g^{\mu_1\mu_2\mu_L\kappa}_{\mu_3\rho} \equiv\; &g^{\mu_1}_{\mu_3}\left(g^{\mu_2\mu_L}g^{\kappa}_{\rho}-g^{\kappa\mu_L}g^{\mu_2}_{\rho}\right)-g^{\mu_2}_{\mu_3}\left(g^{\mu_1\mu_L}g^{\kappa}_{\rho}-g^{\kappa\mu_L}g^{\mu_1}_{\rho}\right)\\ &+g^{\kappa}_{\mu_3}\left(g^{\mu_1\mu_L}g^{\mu_2}_{\rho}-g^{\mu_2\mu_L}g^{\mu_1}_{\rho}\right),\end{aligned} \quad (230)$$

which coincides exactly with Eq. (181). Consequently, the covariant projection tensor formulation in the covariant projection tensor method provides a systematic completion of the Lorentz structures entering the orbital-spin coupling in the covariant tensor method, and it allows the corresponding coupling structures to be computed for arbitrary spin.

However, the covariant projection tensor amplitude in the GS scheme has the same issue as the covariant tensor method, namely that the orbital and spin decomposition is not fully explicit. The reason is that, in the COM frame, the covariant projection tensor amplitude in the GS scheme in Eq. (224) evaluates the final-state external spin wave functions at the COM momenta p_1^* and p_2^*, rather than at the rest frame. It can be written as

$$u(p_3;s_3)\bar{u}(p_1;s_1)\bar{u}(p_2;s_2)\xrightarrow{\text{COM}}u(k_3;s_3)\bar{u}(p_1^*;s_1)\bar{u}(p_2^*;s_2),\tag{231}$$

where the general-spin part in the COM frame depends on the direction of the relative momentum and thus contains residual contributions from the orbital part. Consequently, the separation between L and S does not follow the traditional-LS method, since the general-spin part here is not purely intrinsic-spin objects. In this sense, the covariant projection tensor amplitude in the GS scheme differs from the traditional-LS amplitude in Eq. (70). Only in the non-relativistic limit does the general-spin part reduce to $u(k_3;s_3)\bar{u}(k_1;s_1)\bar{u}(k_2;s_2)$, which are the purely intrinsic-spin objects, and the covariant projection tensor amplitude in the GS scheme would have clear LS separation.

Therefore, in order to construct a scheme with a complete orbital-spin separation while using the covariant LS coupling structures, the external spin wave functions must be further processed. Before introducing this treatment, the full amplitude in the GS scheme is defined by summing the LS partial wave amplitudes over all allowed (L,S) components in the COM definition as

$$C_{\sigma_3}^{\sigma_1\sigma_2}(k_3,p_1^*,p_2^*)=\sum_{LS}N_{LS}C_{\sigma_3}^{\sigma_1\sigma_2}(k_3,p_1^*,p_2^*;L,S),\tag{232}$$

where N_{LS} denotes the partial wave coefficient associated with the (L,S) component. The full covariant projection tensor amplitude in the GS scheme can also be written as a complete contraction between the covariant coupling structure and the general-spin part,

$$\begin{aligned}C_{\sigma_3}^{\sigma_1\sigma_2}(k_3,p_1^*,p_2^*)=&\underbrace{\Gamma_{\ell_3}^{\ell_1\ell_2}(k_3,p_1^*,p_2^*)}_{\text{covariant coupling structure}}\\&\times\underbrace{u_{\sigma_3}^{\ell_3}(k_3;s_3)\bar{u}_{\ell_1}^{\sigma_1}(p_1^*;s_1)\bar{u}_{\ell_2}^{\sigma_2}(p_2^*;s_2)}_{\text{general-spin part}},\end{aligned}\tag{233}$$

with

$$\Gamma_{\ell_3}^{\ell_1\ell_2}(k_3,p_1^*,p_2^*)\equiv\sum_{LS}N_{LS}\Gamma_{\ell_3}^{\ell_1\ell_2}(k_3,p_1^*,p_2^*;L,S).\tag{234}$$

In the next subsubsection, the final-state spin wave functions are rewritten such that their momentum dependence is extracted and reassigned to the coupling structure. This leads to the PS scheme partial wave amplitudes, where the orbital and spin parts are explicitly separated.

2. *Covariant projection tensor method in the PS scheme*

In this subsubsection, the construction of the covariant projection tensor amplitude in the PS scheme is discussed. In Eq. (233), the general-spin part depends on the relative momentum and, therefore, carries orbital information. To enforce a complete separation between the orbital part and the spin part, the orbital dependence in the final-state wave functions is extracted and absorbed into the covariant coupling structure. Using Eq. (192), the final-state spin wave functions can be written as

$$\begin{aligned}\bar{u}_{\ell_1}^{\sigma_1}(p_1^*;s_1)&=\bar{u}_{\ell_1'}^{\sigma_1}(k_1;s_1)D_{\ell_1}^{\ell_1'}(L_{1,c}^{-1}(\mathbf{p}_1^*)),\\\bar{u}_{\ell_2}^{\sigma_2}(p_2^*;s_2)&=\bar{u}_{\ell_2'}^{\sigma_2}(k_2;s_2)D_{\ell_2}^{\ell_2'}(L_{2,c}^{-1}(\mathbf{p}_2^*)),\end{aligned}\tag{235}$$

where k_1 and k_2 are the standard momenta of particle-1 and particle-2. The boosts are then reassigned to the covariant coupling structure. Using Eqs. (233) and (235), the covariant projection tensor amplitude in the PS scheme can be obtained as

$$\begin{aligned}A_{\sigma_3}^{\sigma_1\sigma_2}(k_3,p_1^*,p_2^*)=&\underbrace{\Gamma_{\ell_3}^{\ell_1\ell_2}(k_3,p_1^*,p_2^*)D_{\ell_1}^{\ell_1'}(L_{1,c}^{-1}(\mathbf{p}_1^*))D_{\ell_2}^{\ell_2'}(L_{2,c}^{-1}(\mathbf{p}_2^*))}_{\text{covariant coupling structure}}\\&\times\underbrace{u_{\sigma_3}^{\ell_3}(k_3;s_3)\bar{u}_{\ell_1'}^{\sigma_1}(k_1;s_1)\bar{u}_{\ell_2'}^{\sigma_2}(k_2;s_2)}_{\text{pure-spin part}},\end{aligned}\tag{236}$$

where the pure-spin part no longer contains any orbital information, thereby achieving an explicit separation between the orbital and spin parts.

For the covariant coupling structure in Eq. (236), once all allowed (L,S) partial waves are included, it can be expanded in the set of covariant LS coupling structures $\Gamma_{\ell_3}^{\ell_1\ell_2}(k_3,p_1^*,p_2^*;L,S)$ defined in Eq. (223), which can be written as

$$\begin{aligned}&\Gamma_{\ell_3}^{\ell_1\ell_2}(k_3,p_1^*,p_2^*)D_{\ell_1}^{\ell_1'}(L_{1,c}^{-1}(\mathbf{p}_1^*))D_{\ell_2}^{\ell_2'}(L_{2,c}^{-1}(\mathbf{p}_2^*))\\&=\sum_{LS}N_{LS}\Gamma_{\ell_3}^{\ell_1'\ell_2'}(k_3,p_1^*,p_2^*;L,S),\end{aligned}\tag{237}$$

where N_{LS} is the partial wave coefficient of the LS amplitude basis. As a set of pure tensor structures, $\Gamma_{\ell_3}^{\ell_1\ell_2}(k_3,p_1^*,p_2^*;L,S)$ does not necessarily (and in practice usually does not) form a complete basis. However, after contracting it with the external spin wave functions, one obtains the scalar amplitudes.

$$\Gamma^{\ell_1\ell_2}_{\ell_3}(k_3,p_1^*,p_2^*;L,S)\times u^{\ell_3}_{\sigma_3}(k_3;s_3)\bar{u}^{\sigma_1}_{\ell_1}(k_1;s_1)\bar{u}^{\sigma_2}_{\ell_2}(k_2;s_2), \quad (238)$$

which reproduces the LS decomposition of the traditional-LS amplitude. In this sense, the decomposition is complete at the amplitude level. Therefore, the structures generated by $\Gamma^{\ell_1\ell_2}_{\ell_3}(k_3,p_1^*,p_2^*;L,S)$ provide a complete basis for the amplitudes. Any additional tensor structures can only yield redundant contributions that become equivalent to this basis after contraction with the external spin wave functions, and they do not generate new independent amplitudes. Based on this observation, it is sufficient to consider only the tensor structures generated by these $\Gamma^{\ell_1\ell_2}_{\ell_3}(k_3,p_1^*,p_2^*;L,S)$ to span the full set of amplitudes. Consequently, the covariant projection tensor amplitude in the PS scheme is defined as

$$\begin{aligned}A^{\sigma_1\sigma_2}_{\sigma_3}(k_3,p_1^*,p_2^*;L,S)\equiv\ &\underbrace{\Gamma^{\ell_1\ell_2}_{\ell_3}(k_3,p_1^*,p_2^*;L,S)}_{\text{covariant LS coupling structure}}\\ &\times\underbrace{u^{\ell_3}_{\sigma_3}(k_3;s_3)\bar{u}^{\sigma_1}_{\ell_1}(k_1;s_1)\bar{u}^{\sigma_2}_{\ell_2}(k_2;s_2)}_{\text{pure-spin part}},\end{aligned} \quad (239)$$

where the LS coupling structure $\Gamma^{\ell_1\ell_2}_{\ell_3}(k_3,p_1^*,p_2^*;L,S)$ is the same as those in Eq. (223), while the external spin wave functions are defined at the standard momenta of the corresponding particles.

In the following, the equivalence between the covariant projection tensor amplitude in the PS scheme and the traditional-LS amplitude is established. Using Eqs. (223), (203), (207), and (219), the amplitude in the COM frame can be expanded as

$$\begin{aligned}&A^{\sigma_1\sigma_2}_{\sigma_3}(k_3,p_1^*,p_2^*;L,S)\\ &=\Gamma^{\ell_1\ell_2}_{\ell_3}(k_3,p_1^*,p_2^*;L,S)u^{\ell_3}_{\sigma_3}(k_3;s_3)\bar{u}^{\sigma_1}_{\ell_1}(k_1;s_1)\bar{u}^{\sigma_2}_{\ell_2}(k_2;s_2)\\ &=\sum_{\sigma_1'\sigma_2'\sigma_S'\sigma_L\sigma_S\sigma_3'} C^{L,\sigma_L;S,\sigma_S}_{s_3,\sigma_3'}\bar{u}^{\sigma_3'}_{\ell_3}(k_3;s_3)u^{\ell_L}_{\sigma_L}(k_3;L)u^{\ell_S}_{\sigma_S}(k_3;S)\\ &\times C^{s_1,\sigma_1';s_2,\sigma_2'}_{S,\sigma_S'}\bar{u}^{\sigma_S'}_{\ell_S}(k_3;S)u^{\ell_1}_{\sigma_1'}(k_3;s_1)u^{\ell_2}_{\sigma_2'}(k_3;s_2)\\ &\times P^{\mu_1\cdots\mu_L}_{\ell_L}(k_3;L)(p_1^*-p_2^*)_{\mu_1}\cdots(p_1^*-p_2^*)_{\mu_L}\\ &\times u^{\ell_3}_{\sigma_3}(k_3;s_3)\bar{u}^{\sigma_1}_{\ell_1}(k_1;s_1)\bar{u}^{\sigma_2}_{\ell_2}(k_2;s_2).\end{aligned} \quad (240)$$

Using the orthonormality relation in Eq. (355) for the spin wave functions with spins s_1, s_2, s_3, and S,

$$\begin{aligned}&\bar{u}^{\sigma_1}_{\ell_1}(k_1;s_1)u^{\ell_1}_{\sigma_1'}(k_3;s_1)\propto\delta^{\sigma_1}_{\sigma_1'},\\ &\bar{u}^{\sigma_2}_{\ell_2}(k_2;s_2)u^{\ell_2}_{\sigma_2'}(k_3;s_2)\propto\delta^{\sigma_2}_{\sigma_2'},\\ &\bar{u}^{\sigma_3'}_{\ell_3}(k_3;s_3)u^{\ell_3}_{\sigma_3}(k_3;s_3)=\delta^{\sigma_3}_{\sigma_3'},\\ &\bar{u}^{\sigma_S'}_{\ell_S}(k_3;S)u^{\ell_S}_{\sigma_S}(k_3;S)=\delta^{\sigma_S}_{\sigma_S'},\end{aligned} \quad (241)$$

where the spin-s_1 and spin-s_2 wave functions are defined at different standard momenta, the difference is only an overall mass-dependent constant. This constant can be absorbed into the LS partial wave coefficients. Therefore, the amplitude can be written as

$$\begin{aligned}&A^{\sigma_1\sigma_2}_{\sigma_3}(k_3,p_1^*,p_2^*;L,S)\\ &\propto\sum_{\sigma_L\sigma_S} C^{L,\sigma_L;S,\sigma_S}_{s_3,\sigma_3}C^{s_1,\sigma_1;s_2,\sigma_2}_{S,\sigma_S}\\ &\times u^{\ell_L}_{\sigma_L}(k_3;L)P^{\mu_1\cdots\mu_L}_{\ell_L}(k_3;L)(p_1^*-p_2^*)_{\mu_1}\cdots(p_1^*-p_2^*)_{\mu_L}.\end{aligned} \quad (242)$$

For the expression in the second line, after simplification, it becomes the spherical harmonic with orbital angular momentum L and can be written as

$$u^{\ell_L}_{\sigma_L}(\mathbf{k}_3;L)P^{\mu_1\cdots\mu_L}_{\ell_L}(\mathbf{k}_3;L)(p_1^*-p_2^*)_{\mu_1}\cdots(p_1^*-p_2^*)_{\mu_L}=N_LY^L_{\sigma_L}(\Omega), \quad (243)$$

where N_L is the normalization constant. As an example, for $L=1$,

$$\begin{aligned}&u^{\ell_1}_{\sigma_1}(k_3;1)P^{\mu_1}_{\ell_1}(\mathbf{k}_3;1)(p_1^*-p_2^*)_{\mu_1}\\ &=u^{\ell_1}_{\sigma_1}(k_3;1)P^{\mu_1}_{\ell_1}(k_3;1,1,0)(p_1^*-p_2^*)_{\mu_1}\\ &=\sum_{\sigma_1'\sigma_1''}u^{\ell_1}_{\sigma_1}(k_3;1)C^{1,\sigma_1'';0}_{1,\sigma_1'}\bar{u}^{\sigma_1'}_{\ell_1}(k_3;1)u^{\mu_1}_{\sigma_1''}(k_3;1)(p_1^*-p_2^*)_{\mu_1}\\ &=u^{\mu_1}_{\sigma_1}(k_3;1)(p_1^*-p_2^*)_{\mu_1}.\end{aligned} \quad (244)$$

In the rest frame, the spin-1 wave function with the Lorentz four-vector index μ corresponds to the polarization vector. For the standard momentum k_3^μ, it can be written as

$$u^\mu_\sigma(k_3;1)=\epsilon^\mu_\sigma(k_3)=\begin{pmatrix}0&\frac{-1}{\sqrt{2}}&\frac{-\mathrm{i}}{\sqrt{2}}&0\\0&0&0&1\\0&\frac{1}{\sqrt{2}}&\frac{-\mathrm{i}}{\sqrt{2}}&0\end{pmatrix}^{\mu}_{\sigma},$$

$$\bar{u}^\sigma_\mu(k_3;1)=\epsilon^{*\sigma}_\mu(k_3)=\begin{pmatrix}0&0&0\\\frac{-1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\\frac{\mathrm{i}}{\sqrt{2}}&0&\frac{\mathrm{i}}{\sqrt{2}}\\0&1&0\end{pmatrix}^{\sigma}_{\mu}. \quad (245)$$

By calculating each component of Eq. (244), one finds that the expression is proportional to the spherical harmonics and can be written as

$$u^{\mu_1}_{\sigma_1}(k_3;1)(p_1^*-p_2^*)_{\mu_1} \propto \begin{pmatrix} -\sin\theta e^{i\phi} \\ \cos\theta \\ \sin\theta e^{-i\phi} \end{pmatrix}_{\sigma_1} = Y^1_{\sigma_1}(\Omega). \quad (246)$$

In this way, for orbital angular momentum L, the same simplification can be performed as above, and the result reduces to the spherical harmonic in Eq. (243). The normalization constant N_L can be absorbed into the LS partial wave coefficient. Thus, the amplitude in Eq. (242) can be written as

$$A^{\sigma_1\sigma_2}_{\sigma_3}(k_3,p_1^*,p_2^*;L,S) \propto \sum_{\sigma_L\sigma_S} C^{L,\sigma_L;S,\sigma_S}_{s_3,\sigma_3} C^{s_1,\sigma_1;s_2,\sigma_2}_{S,\sigma_S} Y^L_{\sigma_L}(\Omega). \quad (247)$$

This shows that the covariant projection tensor amplitude in the PS scheme is equivalent to the traditional-LS amplitude in the COM frame. Consequently, the LS expansion of the coupling structure in Eq. (237) is complete once all allowed (L,S) are included. In the PS scheme, the LS partial wave amplitude therefore realizes an explicit separation between the orbital part and the spin part in the COM definition. The same construction is extended below to any frame.

From Eq. (96), the canonical amplitude in any frame can be written as the COM frame canonical amplitude multiplied by the corresponding Wigner rotations of the final-state particles, which can be expressed as

$$\begin{aligned} &A^{\sigma_1\sigma_2}_{\sigma_3}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2;L,S) \\ &= \sum_{\sigma_1'\sigma_2'} A^{\sigma_1'\sigma_2'}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}_1^*,\mathbf{p}_2^*;L,S) D^{\sigma_1'(s_1)*}_{\sigma_1}(R_{31}) D^{\sigma_2'(s_2)*}_{\sigma_2}(R_{32}). \end{aligned} \quad (248)$$

Therefore, in any frame, the covariant projection tensor amplitude in the PS scheme transforms with the same induced little group rotations as the traditional-LS amplitude.

Since the little group is a subgroup of the Lorentz group, its action on the wave functions at the standard momentum can be expressed as a Lorentz transformation implementing the little group rotation on the standard momentum wave functions. Using Eq. (192), the spin wave function under the little group rotation is written as

$$\begin{aligned} &\sum_{\sigma'} D^{\sigma'(s)}_{\sigma}(R) u^{\ell}_{\sigma'}(k;s) = D^{\ell}_{\ell'}(R) u^{\ell'}_{\sigma}(k;s), \\ &\sum_{\sigma'} D^{\sigma(s)}_{\sigma'}(R^{-1}) \bar{u}^{\sigma'}_{\ell}(k;s) = D^{\ell'}_{\ell}(R^{-1}) \bar{u}^{\sigma}_{\ell'}(k;s), \end{aligned} \quad (249)$$

where k is the standard momentum of the particle. Finally, in any frame, the covariant projection tensor amplitude in the PS scheme can be written as

$$\begin{aligned} &A^{\sigma_1\sigma_2}_{\sigma_3}(p_3,p_1,p_2;L,S) \\ &= \Gamma^{\ell_1\ell_2}_{\ell_3}(k_3,p_1^*,p_2^*;L,S) D^{\ell_1'}_{\ell_1}(R_{31}^{-1}) D^{\ell_2'}_{\ell_2}(R_{32}^{-1}) \\ &\quad \times u^{\ell_3}_{\sigma_3}(k_3;s_3) \bar{u}^{\sigma_1}_{\ell_1'}(k_1;s_1) \bar{u}^{\sigma_2}_{\ell_2'}(k_2;s_2), \end{aligned} \quad (250)$$

where R_{3i} $(i=1,2)$ are the Wigner rotations obtained from Eq. (95) and can be written as

$$R_{3i} = L^{-1}_{i,c}(\mathbf{p}_i^*) L^{-1}_{3,c}(\mathbf{p}_3) L_{i,c}(\mathbf{p}_i), \qquad i=1,2, \quad (251)$$

where $L_{i,c}(\mathbf{p}_i)$ is the canonical standard boost for particle-i. The matrix $D^{\ell'}_{\ell}(R)$ denotes the Lorentz transformation matrix in the Lorentz representation $[\ell]$. Although the covariant projection tensor amplitude in the PS scheme can be evaluated in any frame, the associated Wigner rotation depends on the particle momenta $\mathbf{p}_i^*$ $(i=1,2)$ in the COM frame. Therefore, the PS scheme is not fully manifestly covariant at the level of its formal definition.

The covariant projection tensor amplitudes in both the PS scheme and the GS scheme are related by the corresponding Lorentz transformations, which can be written as

$$\begin{aligned} &A^{\sigma_1\sigma_2}_{\sigma_3}(p_3,p_1,p_2;L,S) \\ &= \Gamma^{\ell_1\ell_2}_{\ell_3}(p_3,p_1,p_2;L,S) u^{\ell_3}_{\sigma_3}(p_3;s_3) \bar{u}^{\sigma_1}_{\ell_1'}(p_1;s_1) \bar{u}^{\sigma_2}_{\ell_2'}(p_2;s_2) \\ &\quad \times D^{\ell_1'}_{\ell_1}\left(L_{3,c}(\mathbf{p}_3) R_{31}^{-1} L^{-1}_{3,c}(\mathbf{p}_3)\right) D^{\ell_2'}_{\ell_2}\left(L_{3,c}(\mathbf{p}_3) R_{32}^{-1} L^{-1}_{3,c}(\mathbf{p}_3)\right) \\ &= \Gamma^{\ell_1\ell_2}_{\ell_3}(p_3,p_1,p_2;L,S) u^{\ell_3}_{\sigma_3}(p_3;s_3) \bar{u}^{\sigma_1}_{\ell_1'}(p_1;s_1) \bar{u}^{\sigma_2}_{\ell_2'}(p_2;s_2) \\ &\quad \times D^{\ell_1'}_{\ell_1}\left(L_{3,c}(\mathbf{p}_3) L_{1,c}(\mathbf{p}_1^*) L^{-1}_{3,c}(\mathbf{p}_3)\right) \\ &\quad \times D^{\ell_2'}_{\ell_2}\left(L_{3,c}(\mathbf{p}_3) L_{2,c}(\mathbf{p}_2^*) L^{-1}_{3,c}(\mathbf{p}_3)\right), \end{aligned} \quad (252)$$

where, in the last equality, the first line coincides with the covariant projection tensor amplitude in the GS scheme, while the second line collects the Lorentz representation matrices that relate the PS scheme and GS scheme definitions. In general, these matrices are not the identity matrix, so the two schemes define different (L,S) partial waves and therefore need not coincide.

Although the PS scheme partial wave amplitude achieves the same explicit separation between the orbital part and the spin part as in the traditional-LS construction in the COM frame, it is not manifestly covariant in the strict sense. From the viewpoint of Lorentz transformations, the "covariance" in the PS scheme refers to the fact that no explicit little group rotation appears on the little group indices. The little group rotations induced by a Lorentz transformation are instead mapped, through the use of the rest frame wave functions, into rotations acting on the Lorentz indices of the rest frame wave functions. However, this procedure does not remove the need to evaluate the Wigner rotations R_{3i} generated by boosts. As a result, the PS scheme realizes an explicit orbital-spin separation, but it does not provide a fully manifestly covariant construction.

C. Calculations and comparisons of different *LS* amplitudes

In this subsubsection, we will calculate the amplitudes for the two-body decay process in the traditional-*LS* amplitude, the covariant projection tensor amplitude in the PS scheme, the Zemach tensor amplitude, and the covariant tensor amplitude. In the following calculations, we omit the overall constant factors g_{LS} in front of the corresponding *LS* partial waves, and focus instead on the values of σ_3,σ_1, and σ_2 associated with each amplitude.

Different *LS* amplitude in COM frame In the following discussion of the covariant projection tensor amplitude in the PS scheme, the Lorentz representations are illustrated using Lorentz four-vector indices denoted by $\mu,\nu,\rho\cdots$. In the covariant tensor method, the resulting amplitude is equivalent to the covariant projection tensor amplitude in the GS scheme. Its difference from the PS scheme originates from the choice of external spin wave functions in the COM frame. Therefore, in the explicit examples of the covariant tensor method, the coupling structure can be taken directly from the PS scheme construction.

For the two-body decay process $3\to 1+2$ with spins $s_3=1, s_1=1, s_2=0$, the momenta of the particles are $\{k_3,p_1^*,p_2^*\}$, and we define $r=p_1^*-p_2^*$. In this process, there are three possible *LS* partial waves, namely $(L,S)=(0,1)$, $(1,1)$, and $(2,1)$.

1. For the $(0,1)$ partial wave, the amplitude can be expressed in the following four schemes:

(a) **Traditional-*LS* amplitude**:

$$A^{\sigma_1}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}_1^*,\mathbf{p}_2^*;0,1)=C^{0,\sigma_L;1,\sigma_S}_{1,\sigma_3}C^{1,\sigma_1;0,\sigma_2}_{1,\sigma_S}Y^0_{\sigma_L}(\Omega)=\delta^{\sigma_1}_{\sigma_3}. \tag{253}$$

(b) **The covariant projection tensor amplitude in the PS scheme:**

$$\begin{aligned}&A^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;0,1)\\&=\Gamma^{\mu_1}_{\mu_3}(k_3,p_1^*,p_2^*;0,1)u^{\mu_3}_{\sigma_3}(k_3;1)\bar u^{\sigma_1}_{\mu_1}(k_1;1)\\&=P^{\mu_S}_{\mu_3}(k_3;1,0,1)P^{\mu_1}_{\mu_S}(k_3;1,1,0)u^{\mu_3}_{\sigma_3}(k_3;1)\bar u^{\sigma_1}_{\mu_1}(k_1;1),\end{aligned} \tag{254}$$

where

$$\begin{aligned}P^{\mu_1}_{\mu_S}(k_3;1,1,0)&=\sum_{\sigma_1\sigma_S}C^{\sigma_1;0}_{1,\sigma_S}\bar u^{\sigma_S}_{\mu_S}(k_3;1)u^{\mu_1}_{\sigma_1}(k_3;1)\\&=\sum_{\sigma_1}\epsilon^{\mu_1}_{\sigma_1}(k_3)\epsilon^{*\sigma_1}_{\mu_S}(k_3)\\&=-g^{\mu_1}_{\mu_S}+\frac{k_3^{\mu_1}k_{3\mu_S}}{m_3^2}=\tilde g^{\mu_1}_{\mu_S}(k_3),\end{aligned} \tag{255}$$

and

$$\begin{aligned}P^{\mu_S}_{\mu_3}(k_3;1,0,1)&=\sum_{\sigma_S\sigma_3}C^{0;1,\sigma_S}_{1,\sigma_3}\bar u^{\sigma_3}_{\mu_3}(k_3;1)u^{\mu_S}_{\sigma_S}(k_3;1),\\&=\sum_{\sigma_S}\epsilon^{\mu_S}_{\sigma_S}(k_3)\epsilon^{*\sigma_S}_{\mu_3}(k_3)\\&=-g^{\mu_S}_{\mu_3}+\frac{k_3^{\mu_S}k_{3\mu_3}}{m_3^2}=\tilde g^{\mu_S}_{\mu_3}(k_3).\end{aligned} \tag{256}$$

The amplitude is

$$\begin{aligned}&A^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;0,1)\\&=P^{\mu_S}_{\mu_3}(k_3;1,0,1)P^{\mu_1}_{\mu_S}(k_3;1,1,0)u^{\mu_3}_{3,\sigma_3}(k_3;1)\bar u^{\sigma_1}_{1,\mu_1}(k_1;1),\\&=\tilde g^{\mu_1}_{\mu_3}(k_3)\epsilon^{\mu_3}_{\sigma_3}(k_3)\epsilon^{*\sigma_1}_{\mu_1}(k_1)\\&=\delta^{\sigma_1}_{\sigma_3}.\end{aligned} \tag{257}$$

(c) **The Zemach tensor amplitude**:

$$A^{\sigma_1}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}_1^*,\mathbf{p}_2^*;0,1)=\sum_i(\epsilon_{\sigma_3})_i(\epsilon^{*\sigma_1})_i=\delta^{\sigma_1}_{\sigma_3}. \tag{258}$$

(d) **The covariant tensor amplitude**:

$$C^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;0,1)=\tilde g^{\mu_1}_{\mu_3}(k_3)\epsilon^{\mu_3}_{\sigma_3}(k_3)\epsilon^{*\sigma_1}_{\mu_1}(p_1^*)\neq\delta^{\sigma_1}_{\sigma_3}. \tag{259}$$

2. For the $(1,1)$ partial wave, the amplitude can be written in the following four schemes:

(a) **Traditional-*LS* amplitude**:

$$\begin{aligned}&A^{\sigma_1}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}_1^*,\mathbf{p}_2^*;1,1)\\&=C^{1,\sigma_L;1,\sigma_S}_{1,\sigma_3}C^{1,\sigma_1;0,\sigma_2}_{1,\sigma_S}Y^1_{\sigma_L}(\Omega)\\&=\frac12\sqrt{\frac{3}{2\pi}}\begin{pmatrix}-\cos\theta & -\frac{1}{\sqrt2}\sin\theta e^{i\phi} & 0\\ -\frac{1}{\sqrt2}\sin\theta e^{-i\phi} & 0 & -\frac{1}{\sqrt2}\sin\theta e^{i\phi}\\ 0 & -\frac{1}{\sqrt2}\sin\theta e^{-i\phi} & \cos\theta\end{pmatrix}^{\sigma_1}_{\sigma_3}\end{aligned} \tag{260}$$

(b) **The covariant projection tensor amplitude in the PS scheme**:

$$\begin{aligned}&A^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;1,1)\\&=\Gamma^{\mu_1}_{\mu_3}(k_3,p_1^*,p_2^*;1,1)u^{\mu_3}_{\sigma_3}(k_3;1)\bar u^{\sigma_1}_{\mu_1}(k_1;1)\\&=P^{\mu_L\mu_S}_{\mu_3}(k_3;1,1,1)P^{\mu_1}_{\mu_S}(k_3;1,1,0)t^1_{\mu_L}(k_3,r)u^{\mu_3}_{\sigma_3}(k_3;1)\bar u^{\sigma_1}_{\mu_1}(k_1;1),\end{aligned} \tag{261}$$

where $P^{\mu_1}_{\mu_S}(\mathbf{k}_3;1,1,0)$ can be obtained from Eq. (255).

$$\begin{aligned}
&P^{\mu_L\mu_S}_{\mu_3}(k_3;1,1,1)\\
&=\sum_{\sigma_L\sigma_S\sigma_3} C^{1,\sigma_L;1,\sigma_S}_{1,\sigma_3}\bar{u}^{\sigma_3}_{\mu_3}(k_3;1)u^{\mu_L}_{\sigma_L}(k_3;1)u^{\mu_S}_{\sigma_S}(k_3;1)\\
&=\sum_{\sigma_L\sigma_S\sigma_3} C^{1,\sigma_L;1,\sigma_S}_{1,\sigma_3}\epsilon^{*\sigma_3}_{\mu_3}(k_3)\epsilon^{\mu_L}_{\sigma_L}(k_3)\epsilon^{\mu_S}_{\sigma_S}(k_3)\\
&=\frac{\mathrm{i}}{\sqrt{2}m_3}g_{\mu_3\mu_3'}\varepsilon^{\mu_3'\mu_L\mu_S\rho}k_{3\rho}\\
&=\frac{\mathrm{i}}{\sqrt{2}m_3}g_{\mu_3\mu_3'}\tilde{\varepsilon}^{\mu_3'\mu_L\mu_S}(k_3),
\end{aligned}\tag{262}$$

and

$$t^1_{\mu_L}(k_3,r)=P^{\mu}_{\mu_L}(k_3;1,1,0)r_\mu=\tilde{g}^{\mu}_{\mu_L}(k_3)r_\mu=\tilde{r}_{\mu_L}(k_3).\tag{263}$$

The amplitude is

$$\begin{aligned}
&A^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;1,1)\\
&=P^{\mu_L\mu_S}_{\mu_3}(k_3;1,1,1)P^{\mu_1}_{\mu_S}(k_3;1,1,0)t^1_{\mu_L}(k_3,r)u^{\mu_3}_{\sigma_3}(k_3;1)\bar{u}^{\sigma_1}_{\mu_1}(k_1;1),\\
&=\frac{\mathrm{i}}{\sqrt{2}m_3}g_{\mu_3\mu_3'}\tilde{\varepsilon}^{\mu_3'\mu_L\mu_1}(k_3)\tilde{r}_{\mu_L}(k_3)\epsilon^{\mu_3}_{\sigma_3}(k_3)\epsilon^{*\sigma_1}_{\mu_1}(k_1)\\
&\propto \begin{array}{c|c} & \sigma_1 \\ \hline \sigma_3 & \begin{pmatrix} -\cos\theta & -\frac{1}{\sqrt{2}}\sin\theta e^{\mathrm{i}\phi} & 0\\ -\frac{1}{\sqrt{2}}\sin\theta e^{-\mathrm{i}\phi} & 0 & -\frac{1}{\sqrt{2}}\sin\theta e^{\mathrm{i}\phi}\\ 0 & -\frac{1}{\sqrt{2}}\sin\theta e^{-\mathrm{i}\phi} & \cos\theta \end{pmatrix}\end{array}.
\end{aligned}\tag{264}$$

(c) **The Zemach tensor amplitude**:

$$\begin{aligned}
&A^{\sigma_1}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}_1^*,\mathbf{p}_2^*;1,1)\\
&=\sum_{ijk}(\epsilon_{\sigma_3})_k\varepsilon_{ijk}(\epsilon^{*\sigma_1})_i\mathbf{r}_j\\
&\propto \begin{array}{c|c} & \sigma_1 \\ \hline \sigma_3 & \begin{pmatrix} -\cos\theta & -\frac{1}{\sqrt{2}}\sin\theta e^{\mathrm{i}\phi} & 0\\ -\frac{1}{\sqrt{2}}\sin\theta e^{-\mathrm{i}\phi} & 0 & -\frac{1}{\sqrt{2}}\sin\theta e^{\mathrm{i}\phi}\\ 0 & -\frac{1}{\sqrt{2}}\sin\theta e^{-\mathrm{i}\phi} & \cos\theta \end{pmatrix}\end{array}.
\end{aligned}\tag{265}$$

(d) **The covariant tensor amplitude**:

$$\begin{aligned}
&C^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;1,1)\\
&=\frac{\mathrm{i}}{\sqrt{2}m_3}g_{\mu_3\mu_3'}\tilde{\varepsilon}^{\mu_3'\mu_L\mu_1}(k_3)\tilde{r}_{\mu_L}(k_3)\epsilon^{\mu_3}_{\sigma_3}(k_3)\epsilon^{*\sigma_1}_{\mu_1}(p_1^*)\\
&\not\propto \begin{array}{c|c} & \sigma_1 \\ \hline \sigma_3 & \begin{pmatrix} -\cos\theta & -\frac{1}{\sqrt{2}}\sin\theta e^{\mathrm{i}\phi} & 0\\ -\frac{1}{\sqrt{2}}\sin\theta e^{-\mathrm{i}\phi} & 0 & -\frac{1}{\sqrt{2}}\sin\theta e^{\mathrm{i}\phi}\\ 0 & -\frac{1}{\sqrt{2}}\sin\theta e^{-\mathrm{i}\phi} & \cos\theta \end{pmatrix}\end{array}.
\end{aligned}\tag{266}$$

3. For the $(2,1)$ partial wave, the amplitude can be expressed in the following four schemes:

(a) **Traditional-LS amplitude**:

$$\begin{aligned}
A^{\sigma_1}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}_1^*,\mathbf{p}_2^*;2,1)&=C^{2,\sigma_L;1,\sigma_S}_{1,\sigma_3}C^{1,\sigma_1;0,\sigma_2}_{1,\sigma_S}Y^2_{\sigma_L}(\Omega)\\
&=\begin{array}{c|c} & \sigma_1 \\ \hline \sigma_3 & \begin{pmatrix} \frac{1}{2}(\cos^2\theta-\frac{1}{3}) & \frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{\mathrm{i}\phi} & \frac{1}{2}\sin^2\theta e^{2\mathrm{i}\phi}\\ -\frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{-\mathrm{i}\phi} & -(\cos^2\theta-\frac{1}{3}) & -\frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{\mathrm{i}\phi}\\ \frac{1}{2}\sin^2\theta e^{-2\mathrm{i}\phi} & -\frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{-\mathrm{i}\phi} & \frac{1}{2}(\cos^2\theta-\frac{1}{3}) \end{pmatrix}\end{array}.
\end{aligned}\tag{267}$$

(b) **The covariant projection tensor amplitude in the PS scheme:**

$$\begin{aligned}
&A^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;2,1)\\
&=\Gamma^{\mu_1}_{\mu_3}(k_3,p_1^*,p_2^*;2,1)u^{\mu_3}_{\sigma_3}(k_3;1)\bar{u}^{\sigma_1}_{\mu_1}(k_1;1)\\
&=P^{\mu_L\nu_L,\mu_S}_{\mu_3}(k_3;1,2,1)P^{\mu_1}_{\mu_S}(k_3;1,1,0)\\
&\quad\times t^2_{\mu_L\nu_L}(k_3,r)u^{\mu_3}_{\sigma_3}(k_3;1)\bar{u}^{\sigma_1}_{\mu_1}(k_1;1),
\end{aligned}\tag{268}$$

where $P^{\mu_1}_{\mu_S}(\mathbf{k}_3;1,1,0)$ can be obtained from Eq. (255),

$$\begin{aligned}
&P^{\mu_L\nu_L,\mu_S}_{\mu_3}(k_3;1,2,1)\\
&=\sum_{\sigma_L\sigma_S\sigma_3} C^{2,\sigma_L;1,\sigma_S}_{1,\sigma_3}\bar{u}^{\sigma_3}_{\mu_3}(k_3;1)u^{\mu_L\nu_L}_{\sigma_L}(k_3;2)u^{\mu_S}_{\sigma_S}(k_3;1)\\
&=\sum_{\sigma_L\sigma_S\sigma_3} C^{2,\sigma_L;1,\sigma_S}_{1,\sigma_3}\epsilon^{*\sigma_3}_{\mu_3}(k_3)\epsilon^{\mu_L\nu_L}_{\sigma_L}(k_3)\epsilon^{\mu_S}_{\sigma_S}(k_3)\\
&=\frac{1}{2}\left(\tilde{g}^{\mu_L}_{\mu_3}(k_3)\tilde{g}^{\nu_L\mu_S}(k_3)+\tilde{g}^{\nu_L}_{\mu_3}(k_3)\tilde{g}^{\mu_L\mu_S}(k_3)\right)-\frac{1}{3}\tilde{g}^{\mu_L\nu_L}(k_3)\tilde{g}^{\mu_S}_{\mu_3}(k_3),
\end{aligned}\tag{269}$$

and

$$t^2_{\mu_L\nu_L}(k_3,r) = P^{\mu_1\mu_2}_{\mu_L\nu_L}(k_3)r_{\mu_1}r_{\mu_2} = \tilde{r}_{\mu_L}(k_3)\tilde{r}_{\nu_L}(k_3) - \frac{1}{3}(\tilde{r}\cdot\tilde{r})\tilde{g}_{\mu_L\nu_L}(k_3). \tag{270}$$

The amplitude is

$$\begin{aligned}
A^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;2,1) &= P^{\mu_L\nu_L,\mu_S}_{\mu_3}(k_3;1,2,1)P^{\mu_1}_{\mu_S}(k_3;1,1,0)t^2_{\mu_L\nu_L}(k_3,r)u^{\mu_3}_{\sigma_3}(k_3;1)\bar{u}^{\sigma_1}_{\mu_1}(k_1;1)\\
&= \epsilon^{\mu_3}_{\sigma_3}(k_3)\epsilon^{*\sigma_1}_{\mu_1}(k_1)\Big(\tilde{r}^{\mu_1}(k_3)\tilde{r}_{\mu_3}(k_3) - \frac{1}{3}(\tilde{r}\cdot\tilde{r})\tilde{g}^{\mu_1}_{\mu_3}(k_3)\Big)\\
&\propto \begin{pmatrix} \frac{1}{2}(\cos^2\theta-\frac{1}{3}) & \frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{i\phi} & \frac{1}{2}\sin^2\theta e^{2i\phi}\\ -\frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{-i\phi} & -(\cos^2\theta-\frac{1}{3}) & -\frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{i\phi}\\ \frac{1}{2}\sin^2\theta e^{-2i\phi} & -\frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{-i\phi} & \frac{1}{2}(\cos^2\theta-\frac{1}{3})\end{pmatrix}^{\sigma_1}_{\sigma_3}.
\end{aligned} \tag{271}$$

(c) **The Zemach tensor amplitude**:

$$\begin{aligned}
A^{\sigma_1}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}_1^*,\mathbf{p}_2^*;2,1) &= \sum_{ij}(\epsilon_{\sigma_3})_j(\epsilon^{*\sigma_1})_i(\mathbf{r}_i\mathbf{r}_j - \frac{1}{3}\mathbf{r}^2\delta_{ij})\\
&\propto \begin{pmatrix} \frac{1}{2}(\cos^2\theta-\frac{1}{3}) & \frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{i\phi} & \frac{1}{2}\sin^2\theta e^{2i\phi}\\ -\frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{-i\phi} & -(\cos^2\theta-\frac{1}{3}) & -\frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{i\phi}\\ \frac{1}{2}\sin^2\theta e^{-2i\phi} & -\frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{-i\phi} & \frac{1}{2}(\cos^2\theta-\frac{1}{3})\end{pmatrix}^{\sigma_1}_{\sigma_3}.
\end{aligned} \tag{272}$$

(d) **The covariant tensor amplitude**:

$$\begin{aligned}
C^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;2,1) &= \epsilon^{\mu_3}_{\sigma_3}(k_3)\epsilon^{*\sigma_1}_{\mu_1}(p_1^*)\Big(\tilde{r}^{\mu_1}(k_3)\tilde{r}_{\mu_3}(k_3) - \frac{1}{3}(\tilde{r}\cdot\tilde{r})\tilde{g}^{\mu_1}_{\mu_3}(k_3)\Big)\\
&\not\propto \begin{pmatrix} \frac{1}{2}(\cos^2\theta-\frac{1}{3}) & \frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{i\phi} & \frac{1}{2}\sin^2\theta e^{2i\phi}\\ -\frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{-i\phi} & -(\cos^2\theta-\frac{1}{3}) & -\frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{i\phi}\\ \frac{1}{2}\sin^2\theta e^{-2i\phi} & -\frac{1}{\sqrt{2}}\cos\theta\sin\theta e^{-i\phi} & \frac{1}{2}(\cos^2\theta-\frac{1}{3})\end{pmatrix}^{\sigma_1}_{\sigma_3}.
\end{aligned} \tag{273}$$

For the two-body decay process $3\to 1+2$ with spins $s_3=\frac{1}{2}, s_1=\frac{1}{2}, s_2=0$, two possible LS partial waves exist, namely $(L,S)=(0,\frac{1}{2})$ and $(1,\frac{1}{2})$.

1. For the $(0,\frac{1}{2})$ partial wave, the amplitude can be expressed using the following four schemes:

(a) **Traditional-LS amplitude**:

$$A^{\sigma_1}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}_1^*,\mathbf{p}_2^*;0,\frac{1}{2}) = C^{0,\sigma_L;\frac{1}{2},\sigma_S}_{\frac{1}{2},\sigma_3}C^{\frac{1}{2},\sigma_1;0,\sigma_2}_{\frac{1}{2},\sigma_S}Y^0_{\sigma_L}(\Omega) = \delta^{\sigma_1}_{\sigma_3}. \tag{274}$$

(b) **The covariant projection tensor amplitude in the PS scheme:**

$$\begin{aligned}
&A^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;0,\frac{1}{2})\\
&= \Gamma^{\alpha_1}_{\alpha_3}(k_3,p_1^*,p_2^*;0,\frac{1}{2})u^{\alpha_3}_{\sigma_3}(k_3;\frac{1}{2})\bar{u}^{\sigma_1}_{\alpha_1}(k_1;\frac{1}{2})\\
&= P^{\alpha_S}_{\alpha_3}(k_3;\frac{1}{2},0,\frac{1}{2})P^{\alpha_1}_{\alpha_S}(k_3;\frac{1}{2},\frac{1}{2},0)u^{\alpha_3}_{\sigma_3}(k_3;\frac{1}{2})\bar{u}^{\sigma_1}_{\alpha_1}(k_1;\frac{1}{2}),
\end{aligned} \tag{275}$$

where

$$\begin{aligned}
P^{\alpha_S}_{\alpha_3}(k_3;\frac{1}{2},0,\frac{1}{2}) &= \sum_{\sigma_3\sigma_S}C^{0;\frac{1}{2},\sigma_S}_{\frac{1}{2},\sigma_3}\bar{u}^{\sigma_3}_{\alpha_3}(k_3;\frac{1}{2})u^{\alpha_S}_{\sigma_S}(k_3;\frac{1}{2})\\
&= \delta^{\alpha_S}_{\alpha_3},
\end{aligned} \tag{276}$$

and

$$P^{\alpha_1}_{\alpha_S}(k_3;\frac{1}{2},\frac{1}{2},0)=\sum_{\sigma_S\sigma_1}C^{\frac{1}{2},\sigma_1;0}_{\frac{1}{2},\sigma_S}\bar{u}^{\sigma_S}_{\alpha_S}(k_3;\frac{1}{2})u^{\alpha_1}_{\sigma_1}(k_3;\frac{1}{2})$$
$$=\delta^{\alpha_1}_{\alpha_S}. \tag{277}$$

The amplitude is

$$\begin{aligned}&A^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;0,\frac{1}{2})\\&=P^{\alpha_S}_{\alpha_3}(k_3;\frac{1}{2},0,\frac{1}{2})P^{\alpha_1}_{\alpha_S}(k_3;\frac{1}{2},\frac{1}{2},0)u^{\alpha_3}_{\sigma_3}(k_3;\frac{1}{2})\bar{u}^{\sigma_1}_{\alpha_1}(k_1;\frac{1}{2})\\&=\delta^{\alpha_S}_{\alpha_3}\delta^{\alpha_1}_{\alpha_S}u^{\alpha_3}_{\sigma_3}(k_3;\frac{1}{2})\bar{u}^{\sigma_1}_{\alpha_1}(k_1;\frac{1}{2})\\&\propto\delta^{\sigma_1}_{\sigma_3}.\end{aligned} \tag{278}$$

(c) **The Zemach tensor amplitude**:

$$A^{\sigma_1}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}_1^*,\mathbf{p}_2^*;0,\frac{1}{2})=(\chi_{\sigma_3})^{\alpha}(\chi^{*\sigma_1})_{\alpha}$$
$$=\delta^{\sigma_1}_{\sigma_3}. \tag{279}$$

(d) **The covariant tensor amplitude**:

$$C^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;0,\frac{1}{2})=\delta^{\alpha_S}_{\alpha_3}\delta^{\alpha_1}_{\alpha_S}u^{\alpha_3}_{\sigma_3}(k_3;\frac{1}{2})\bar{u}^{\sigma_1}_{\alpha_1}(p_1^*;\frac{1}{2})$$
$$\neq\delta^{\sigma_1}_{\sigma_3}. \tag{280}$$

2. For the $(1,\frac{1}{2})$ partial wave, the amplitude can be written in the following four schemes:

(a) **Traditional-LS amplitude**:

$$A^{\sigma_1}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}_1^*,\mathbf{p}_2^*;1,\frac{1}{2})=C^{1,\sigma_L;\frac{1}{2},\sigma_S}_{\frac{1}{2},\sigma_3}C^{\frac{1}{2},\sigma_1;0,\sigma_2}_{\frac{1}{2},\sigma_S}Y^1_{\sigma_L}(\Omega)$$
$$=\frac{1}{2}\sqrt{\frac{1}{\pi}}\begin{pmatrix}-\cos\theta & -\sin\theta e^{i\phi}\\ -\sin\theta e^{-i\phi} & \cos\theta\end{pmatrix}^{\sigma_1}_{\sigma_3}. \tag{281}$$

(b) **The covariant projection tensor amplitude in the PS scheme**:

$$\begin{aligned}&A^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;1,\frac{1}{2})\\&=\Gamma^{\alpha_1}_{\alpha_3}(k_3,p_1^*,p_2^*;1,\frac{1}{2})u^{\alpha_3}_{\sigma_3}(k_3;\frac{1}{2})\bar{u}^{\sigma_1}_{\alpha_1}(k_1;\frac{1}{2})\\&=P^{\mu_L\alpha_S}_{\alpha_3}(k_3;\frac{1}{2},1,\frac{1}{2})P^{\alpha_1}_{\alpha_S}(k_3;\frac{1}{2},\frac{1}{2},0)\\&\quad\times t^1_{\mu_L}(k_3,r)u^{\alpha_3}_{\sigma_3}(k_3;\frac{1}{2})\bar{u}^{\sigma_1}_{\alpha_1}(k_1;\frac{1}{2}),\end{aligned} \tag{282}$$

where $P^{\alpha_1}_{\alpha_S}(\mathbf{k}_3;\frac{1}{2},\frac{1}{2},0)$ and $t^1_{\mu_L}(\mathbf{k}_3,\mathbf{r})$ can be obtained from Eqs. (277) and (263),

$$\begin{aligned}&P^{\mu_L\alpha_S}_{\alpha_3}(k_3;\frac{1}{2},1,\frac{1}{2})\\&=\sum_{\sigma_3\sigma_L\sigma_S}C^{1,\sigma_L;\frac{1}{2},\sigma_S}_{\frac{1}{2},\sigma_3}\bar{u}^{\sigma_3}_{\alpha_3}(k_3;\frac{1}{2})u^{\mu_L}_{\sigma_L}(k_3;1)u^{\alpha_S}_{\sigma_S}(k_3;\frac{1}{2})\\&=(\gamma_5\tilde{g}^{\mu_L}_{\mu_L'}(k_3)\gamma^{\mu_L'})^{\alpha_S}_{\alpha_3}.\end{aligned} \tag{283}$$

The amplitude is

$$\begin{aligned}&A^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;1,\frac{1}{2})\\&=P^{\mu_L\alpha_S}_{\alpha_3}(k_3;\frac{1}{2},1,\frac{1}{2})P^{\alpha_1}_{\alpha_S}(k_3;\frac{1}{2},\frac{1}{2},0)\\&\quad\times t^1_{\mu_L}(k_3,r)u^{\alpha_3}_{\sigma_3}(k_3;\frac{1}{2})\bar{u}^{\sigma_1}_{\alpha_1}(k_1;\frac{1}{2})\\&=\bar{u}^{\sigma_1}_{\alpha_1}(k_1;\frac{1}{2})(\gamma_5\tilde{g}^{\mu_L}_{\mu_L'}(k_3)\gamma^{\mu_L'})^{\alpha_1}_{\alpha_3}u^{\alpha_3}_{\sigma_3}(k_3;\frac{1}{2})\tilde{r}_{\mu_L}(k_3)\\&\propto\begin{pmatrix}-\cos\theta & -\sin\theta e^{i\phi}\\ -\sin\theta e^{-i\phi} & \cos\theta\end{pmatrix}^{\sigma_1}_{\sigma_3}.\end{aligned} \tag{284}$$

(c) **The Zemach tensor amplitude**:

$$A^{\sigma_1}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}_1^*,\mathbf{p}_2^*;1,\frac{1}{2})=\sum_i(\chi^{*\sigma_1})_{\alpha}(\sigma_i)^{\alpha}_{\beta}(\chi_{\sigma_3})^{\beta}\mathbf{r}_i$$
$$\propto\begin{pmatrix}-\cos\theta & -\sin\theta e^{i\phi}\\ -\sin\theta e^{-i\phi} & \cos\theta\end{pmatrix}^{\sigma_1}_{\sigma_3}. \tag{285}$$

(d) **The covariant tensor amplitude**:

$$\begin{aligned}&C^{\sigma_1}_{\sigma_3}(k_3,p_1^*,p_2^*;1,\frac{1}{2})\\&=\bar{u}^{\sigma_1}_{\alpha_1}(p_1^*;\frac{1}{2})(\gamma_5\tilde{g}^{\mu_L}_{\mu_L'}(k_3)\gamma^{\mu_L'})^{\alpha_1}_{\alpha_3}u^{\alpha_3}_{\sigma_3}(k_3;\frac{1}{2})\tilde{r}_{\mu_L}(k_3)\\&\not\propto\begin{pmatrix}-\cos\theta & -\sin\theta e^{i\phi}\\ -\sin\theta e^{-i\phi} & \cos\theta\end{pmatrix}^{\sigma_1}_{\sigma_3}.\end{aligned} \tag{286}$$

Through the above comparison, we note that for the traditional-LS amplitudes, the covariant projection tensor amplitude in the PS scheme, and the Zemach tensor amplitudes, they are completely equivalent in the COM frame and use the same LS component.

By contrast, the covariant tensor amplitudes are not equivalent to these amplitudes in the COM frame. Only in the non-relativistic limit are the covariant tensor amplitudes equivalent to the other three amplitudes.

Different LS amplitude in any frame Since the traditional-LS amplitudes, the covariant projection tensor amplitudes in the PS scheme, and the Zemach tensor amplitudes have been shown to be equivalent in the COM frame, boosting them to any frame induces the same little group rotations, which can be written as

$$R_{31} = L^{-1}_{1,c}(\mathbf{p}_1^*)L^{-1}_{3,c}(\mathbf{p}_3)L_{1,c}(\mathbf{p}_1), \tag{287}$$

where only the rotation for particle-1 is written explicitly, since in the examples below, particle-1 is the only final-state particle that carries spin. In the following, the covariant projection tensor amplitudes in the PS scheme are therefore used as the representative scheme for comparisons with the covariant projection tensor amplitudes in the GS scheme in any frame.

For the two-body decay process $3\to 1+2$ with spins $s_3=1$, $s_1=1$, $s_2=0$, the momenta of the particles are $\{p_3,p_1,p_2\}$. In this process, there are three possible LS partial waves, namely $(L,S)=(0,1)$, $(1,1)$, and $(2,1)$.

1. For the (0,1) partial wave, the amplitudes can be written in the following two schemes:

(a) **The covariant projection tensor amplitude in the PS scheme:**

$$\begin{aligned}&A^{\sigma_1}_{\sigma_3}(p_3,p_1,p_2;0,1)\\&=\tilde{g}^{\mu_1}_{\mu_3}(k_3)\epsilon^{\mu_3}_{\sigma_3}(k_3)\epsilon^{*\sigma_1}_{\mu_1'}(k_1)D^{\mu_1'}_{\mu_1}(R^{-1}_{31})\\&=\tilde{g}^{\mu_1}_{\mu_3}(p_3)\epsilon^{\mu_3}_{\sigma_3}(p_3)\epsilon^{*\sigma_1}_{\mu_1'}(k_1)D^{\mu_1'}_{\mu_1}(R^{-1}_{31}L^{-1}_{3,c}(\mathbf{p}_3))\\&=\tilde{g}^{\mu_1}_{\mu_3}(p_3)\epsilon^{\mu_3}_{\sigma_3}(p_3)\epsilon^{*\sigma_1}_{\mu_1'}(p_1)D^{\mu_1'}_{\mu_1}(L_{3,c}(\mathbf{p}_3)L_{1,c}(\mathbf{p}_1^*)L^{-1}_{3,c}(\mathbf{p}_3)).\end{aligned} \tag{288}$$

(b) **The covariant projection tensor amplitude in the GS scheme**:

$$C^{\sigma_1}_{\sigma_3}(p_3,p_2,p_1;0,1)=\tilde{g}^{\mu_1}_{\mu_3}(p_3)\epsilon^{\mu_3}_{\sigma_3}(p_3)\epsilon^{*\sigma_1}_{\mu_1}(p_1). \tag{289}$$

2. For the $(1,1)$ partial wave, the amplitudes can be expressed in the following two schemes:

(a) **The covariant projection tensor amplitude in the PS scheme**:

$$\begin{aligned}&A^{\sigma_1}_{\sigma_3}(p_3,p_1,p_2;1,1)\\&=\frac{\mathrm{i}}{\sqrt{2}m_3}g_{\mu_3\mu_3'}\tilde{\varepsilon}^{\mu_3'\mu_L\mu_1}(p_3)\tilde{r}_{\mu_L}(p_3)\epsilon^{\mu_3}_{\sigma_3}(p_3)\epsilon^{*\sigma_1}_{\mu_1'}(p_1)\\&\quad\times D^{\mu_1'}_{\mu_1}(L_{3,c}(\mathbf{p}_3)L_{1,c}(\mathbf{p}_1^*)L^{-1}_{3,c}(\mathbf{p}_3)).\end{aligned} \tag{290}$$

(b) **The covariant projection tensor amplitude in the GS scheme:**

$$\begin{aligned}&C^{\sigma_1}_{\sigma_3}(p_3,p_2,p_1;1,1)\\&=\frac{\mathrm{i}}{\sqrt{2}m_3}g_{\mu_3\mu_3'}\tilde{\varepsilon}^{\mu_3'\mu_L\mu_1}(p_3)\tilde{r}_{\mu_L}(p_3)\epsilon^{\mu_3}_{\sigma_3}(p_3)\epsilon^{*\sigma_1}_{\mu_1}(p_1).\end{aligned} \tag{291}$$

3. For the $(2,1)$ partial wave, the amplitudes can be written in the following two schemes:

(a) **The covariant projection tensor amplitude in the PS scheme:**

$$\begin{aligned}&A^{\sigma_1}_{\sigma_3}(p_3,p_1,p_2;2,1)\\&=\epsilon^{\mu_3}_{\sigma_3}(p_3)\epsilon^{*\sigma_1}_{\mu_1'}(p_1)\Big(\tilde{r}^{\mu_1}(p_3)\tilde{r}_{\mu_3}(p_3)-\frac{1}{3}(\tilde{r}\cdot\tilde{r})\tilde{g}^{\mu_1}_{\mu_3}(p_3)\Big)\\&\quad\times D^{\mu_1'}_{\mu_1}(L_{3,c}(\mathbf{p}_3)L_{1,c}(\mathbf{p}_1^*)L^{-1}_{3,c}(\mathbf{p}_3)).\end{aligned} \tag{292}$$

(b) **The covariant projection tensor amplitude in the GS scheme**:

$$\begin{aligned}&C^{\sigma_1}_{\sigma_3}(p_3,p_2,p_1;2,1)\\&=\epsilon^{\mu_3}_{\sigma_3}(p_3)\epsilon^{*\sigma_1}_{\mu_1}(p_1)\Big(\tilde{r}^{\mu_1}(p_3)\tilde{r}_{\mu_3}(p_3)-\frac{1}{3}(\tilde{r}\cdot\tilde{r})\tilde{g}^{\mu_1}_{\mu_3}(p_3)\Big).\end{aligned} \tag{293}$$

In the two-body decay process $3\to 1+2$ with spins $s_3=\dfrac{1}{2}, s_1=\dfrac{1}{2}, s_2=0$, there are two possible LS partial waves: namely, $(L,S)=(0,\dfrac{1}{2})$ and $(1,\dfrac{1}{2})$.

1. For the $(0,\frac{1}{2})$ partial wave, the amplitudes can be written in the following two schemes:

(a) **The covariant projection tensor amplitude in the PS scheme:**

$$\begin{aligned}A^{\sigma_1}_{\sigma_3}(p_3,p_1,p_2;0,\frac{1}{2})&=\delta^{\alpha_S}_{\alpha_3}\delta^{\alpha_1}_{\alpha_S}u^{\alpha_3}_{\sigma_3}(p_3;\frac{1}{2})\bar{u}^{\sigma_1}_{\alpha_1'}(p_1;\frac{1}{2})\\&\quad\times D^{\alpha_1'}_{\alpha_1}(L_{3,c}(\mathbf{p}_3)L_{1,c}(\mathbf{p}_1^*)L^{-1}_{3,c}(\mathbf{p}_3)).\end{aligned} \tag{294}$$

(b) **The covariant projection tensor amplitude in the GS scheme:**

$$C^{\sigma_1}_{\sigma_3}(p_3,p_2,p_1;0,\frac{1}{2})=\delta^{\alpha_S}_{\alpha_3}\delta^{\alpha_1}_{\alpha_S}u^{\alpha_3}_{\sigma_3}(p_3;\frac{1}{2})\bar{u}^{\sigma_1}_{\alpha_1}(p_1;\frac{1}{2}). \tag{295}$$

2. For the $(1,\dfrac{1}{2})$ partial wave, the amplitudes can be expressed in the following two schemes:

(a) **The covariant projection tensor amplitude in the PS scheme**:

$$\begin{aligned}&A^{\sigma_1}_{\sigma_3}(p_3,p_1,p_2;1,\frac{1}{2})\\&=\bar{u}^{\sigma_1}_{\alpha_1'}(p_1;\frac{1}{2})(\gamma_5\tilde{g}^{\mu_L}_{\mu_L'}(p_3)\gamma^{\mu_L'})^{\alpha_1}_{\alpha_3}u^{\alpha_3}_{\sigma_3}(p_3;\frac{1}{2})\tilde{r}_{\mu_L}(p_3)\\&\quad\times D^{\alpha_1'}_{\alpha_1}(L_{3,c}(\mathbf{p}_3)L_{1,c}(\mathbf{p}_1^*)L^{-1}_{3,c}(\mathbf{p}_3)).\end{aligned} \tag{296}$$

(b) **The covariant projection tensor amplitude in the GS scheme:**

$$C^{\sigma_1}_{\sigma_3}(p_3,p_2,p_1;1,\frac{1}{2})$$
$$=\bar{u}^{\sigma_1}_{\alpha_1}(p_1;\frac{1}{2})(\gamma_5\tilde{g}^{\mu_L}_{\mu'_L}(p_3)\gamma^{\mu'_L})^{\alpha_1}_{\alpha_3}u^{\alpha_3}_{\sigma_3}(p_3;\frac{1}{2})\tilde{r}_{\mu_L}(p_3). \quad (297)$$

By comparing the two expressions, one can see that the covariant projection tensor amplitude in the PS scheme contains an additional Lorentz transformation factor relative to the covariant projection tensor amplitude in the GS scheme, namely $L_{3,c}(\mathbf{p}_3)L_{1,c}(\mathbf{p}^*_1)L^{-1}_{3,c}(\mathbf{p}_3)$. This factor is generally not the identity matrix, so the (L,S) partial wave bases in the two constructions do not match term by term, and the extracted LS partial wave coefficients therefore need not be the same.

IV. ON-SHELL CONSTRUCTION OF COVARIANT ORBITAL-SPIN COUPLING AMPLITUDE

In the previous section, methods for constructing LS amplitudes using tensors were presented, and each of them has certain limitations. For example, the covariant projection tensor method in the PS scheme, although the covariant projection tensor amplitudes in the PS scheme can be evaluated in any frame, their expressions still involve momenta defined in the COM frame. As a result, the formulation is not fully covariant in form. For the covariant tensor method (GS scheme), although the covariant tensor amplitudes retain a manifestly Lorentz covariant form, the general-spin part carries orbital information when one returns to the COM frame, and therefore the traditional separation between the orbital part and the spin part is not realized. Only in the non-relativistic limit does it agree with the traditional-LS method.

Moreover, when the process involves massless gauge bosons, the use of polarization vectors or similar spin wave functions typically requires gauge invariance constraints to be imposed explicitly. One must then choose a complete set of vertex structures that satisfy these constraints and are linearly independent, in order to avoid gauge redundancy and redundant fit parameters, which makes practical implementations more cumbersome.

To address the issues of the methods discussed above, we introduce our covariant LS amplitudes constructed in the covariant canonical-spinor scheme. They admit a proper separation between orbital and spin parts, do not require boosting to the COM frame, and can be applied directly to processes involving massless particles.

In subsection IV.A, the covariant canonical-spinor scheme is presented. In subsection IV..B, we further compare the covariant canonical-spinor scheme with the covariant projection tensor method in the PS scheme, and reformulate their expressions using our spinor construction to obtain a genuinely covariant form. In subsection IV.C, explicit examples of covariant canonical-spinor amplitude calculations are given.

A. Spinorial construction

In the spinorial construction, spinor variables are taken as the basic building blocks. Before introducing these spinor variables, it is useful to begin with the homogeneous Lorentz group. The six generators of homogeneous Lorentz transformations in Eq. (14) can be combined as

$$A_i=J_i+iK_i,\quad B_i=J_i-iK_i,\qquad (i=1,2,3). \quad (298)$$

They satisfy the following commutation relations:

$$[A_i,A_j]=i\varepsilon_{ijk}A_k,\quad [B_i,B_j]=i\varepsilon_{ijk}B_k,\quad [A_i,B_j]=0. \quad (299)$$

These commutation relations imply that $SO(3,1)\simeq SU(2)_L\otimes SU(2)_R$. Correspondingly, the momentum in the Lorentz four-vector representation $p^\mu=(E,\mathbf{p})$ can be described as bispinors $p_{\alpha\dot{\alpha}}$. The corresponding conversion matrix is $\sigma_{\mu\alpha\dot{\alpha}}$, which can be written as

$$\sigma_{0\alpha\dot{\alpha}}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_{1\alpha\dot{\alpha}}=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$
$$\sigma_{2\alpha\dot{\alpha}}=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix},\quad \sigma_{3\alpha\dot{\alpha}}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \quad (300)$$

Using this matrix on p^μ gives the 2×2 matrix $p_{\alpha\dot{\alpha}}$, which can be written explicitly as

$$p_{\alpha\dot{\alpha}}=p^\mu\sigma_{\mu\alpha\dot{\alpha}}=\begin{pmatrix}p^0+p^3&p^1-\mathrm{i}p^2\\p^1+\mathrm{i}p^2&p^0-p^3\end{pmatrix}. \quad (301)$$

Using the on-shell condition, one has $\det p_{\alpha\dot{\alpha}}=m^2$. Therefore, there are two cases, corresponding to massless and massive particles.

For a massless particle, satisfying $\det p_{\alpha\dot{\alpha}}=0$, the matrix $p_{\alpha\dot{\alpha}}$ has rank one. Thus, it can be decomposed as the direct product of two spinors, λ and $\tilde{\lambda}$, as

$$p_{\alpha\dot{\alpha}}=\lambda_\alpha\tilde{\lambda}_{\dot{\alpha}}. \quad (302)$$

For a massive particle, satisfying $\det p_{\alpha\dot{\alpha}}=m^2$, the matrix $p_{\alpha\dot{\alpha}}$ has rank two instead of one. Thus, it can be decomposed as the sum of two rank-one bispinors as

$$p_{\alpha\dot{\alpha}}=\lambda_{\alpha I}\tilde{\lambda}^I_{\dot{\alpha}}, \quad (303)$$

where $\lambda_{\alpha I}$ and $\tilde{\lambda}^I_{\dot\alpha}$ are spinors carrying the $SU(2)$ little group index $I=\frac{1}{2},-\frac{1}{2}$ and the SL$(2,\mathbb{C})$ Lorentz group index $\alpha/\dot\alpha=1,2$. In this work, the discussion mainly focuses on spinors of massive particles.

Under a Lorentz transformation, the spinor variables transform under the SL$(2,\mathbb{C})$ Lorentz group and the $SU(2)$ little group. The explicit form of the transformation is

$$\begin{aligned}\lambda_{\alpha I}(p)&\to\lambda_{\alpha I}(\Lambda p)=W_I^J\Lambda_\alpha^\beta\lambda_{\beta J}(p),\\ \tilde{\lambda}^I_{\dot\alpha}(p)&\to\tilde{\lambda}^I_{\dot\alpha}(\Lambda p)=(W^{-1})^I_J\tilde{\Lambda}^{\dot\beta}_{\dot\alpha}\tilde{\lambda}^J_{\dot\beta}(p),\end{aligned}\tag{304}$$

where W and W^{-1} are $SU(2)$ matrices, and Λ and $\tilde\Lambda$ are SL$(2,\mathbb{C})$ matrices. The antisymmetric tensor $\varepsilon_{\alpha\beta}=-\varepsilon^{\alpha\beta}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ is used to raise and lower SL$(2,\mathbb{C})$ and $SU(2)$ spinor indices:

$$\psi^\alpha=\varepsilon^{\alpha\beta}\psi_\beta,\quad \psi_\alpha=\psi^\beta\varepsilon_{\alpha\beta},\quad \varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma}=\delta^\alpha_\gamma.\tag{305}$$

To obtain the explicit form of the spinor variables at a general momentum p, one can boost the spinors defined at the standard momentum. The standard momentum can be written in bispinor form as

$$k_{\alpha\dot\alpha}=k^\mu\sigma_{\mu\alpha\dot\alpha}=\lambda_{\alpha I}(k)\tilde{\lambda}^I_{\dot\alpha}(k)=\begin{pmatrix}m&0\\0&m\end{pmatrix}.\tag{306}$$

In general, there are many equivalent spinor solutions; here, a particular one is chosen, given by

$$\lambda_{\alpha I}(k)=\begin{pmatrix}\sqrt{m}&0\\0&\sqrt{m}\end{pmatrix},\quad \tilde{\lambda}^I_{\dot\alpha}(k)=\begin{pmatrix}\sqrt{m}&0\\0&\sqrt{m}\end{pmatrix},\tag{307}$$

which is consistent with the conventional choice. By applying a Lorentz transformation, the standard momentum k is mapped to a general momentum p, which, in two-component form, can be written as

$$p_{\alpha\dot\alpha}=L(\mathbf{p})_\alpha^\beta\tilde{L}(\mathbf{p})_{\dot\alpha}^{\dot\beta}k_{\beta\dot\beta},\tag{308}$$

where $L(\mathbf{p})_\alpha^\beta$ and $\tilde{L}(\mathbf{p})_{\dot\alpha}^{\dot\beta}$ are SL$(2,\mathbb{C})$ Lorentz boosts. The corresponding transformation of the spinor variables is then given by

$$\lambda_{\alpha I}(p)=L(\mathbf{p})_\alpha^\beta\lambda_{\beta I}(k),\quad \tilde{\lambda}^I_{\dot\alpha}(p)=\tilde{L}(\mathbf{p})_{\dot\alpha}^{\dot\beta}\tilde{\lambda}^I_{\dot\beta}(k).\tag{309}$$

Since subsubsection II.A.1 discusses both the helicity and canonical types of boosts, it is natural to introduce the corresponding spinor-helicity variables and canonical-spinor variables generated by these two types of boosts, respectively. From Eqs. (18) and (19), it follows that both types of boosts can be factorized into the product of a pure boost along the z-axis and a rotation. In the left-handed representation, the two-dimensional rotation and the pure boost along the z-axis can be written as

$$\begin{aligned}&R_z=\begin{pmatrix}e^{-i\frac{\phi}{2}}&0\\0&e^{i\frac{\phi}{2}}\end{pmatrix},\ R_y=\begin{pmatrix}\cos\frac{\theta}{2}&-\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2}&\cos\frac{\theta}{2}\end{pmatrix},\\ &B_z=\begin{pmatrix}e^{\frac{\eta}{2}}&0\\0&e^{-\frac{\eta}{2}}\end{pmatrix},\end{aligned}\tag{310}$$

where $E+|\mathbf{p}|=me^\eta$ and $\eta=\text{arctanh}\left(\frac{|\mathbf{p}|}{E}\right)$. For the rotation part of the boost, we take

$$\begin{aligned}R(\phi,\theta,-\phi)&=R_z(\phi)R_y(\theta)R_z(-\phi)\\&=\begin{pmatrix}c&-s^*\\s&c\end{pmatrix},\end{aligned}\tag{311}$$

where $c\equiv\cos\frac{\theta}{2}$ and $s\equiv\sin\frac{\theta}{2}e^{i\phi}$. Here, it should be noted that the choice of rotation $R(\phi,\theta,-\phi)$ differs from the rotation $R(\phi,\theta,0)$ in Eq. (18). Specifically, the first rotation about the z-axis is taken to be $-\phi$. For both the helicity boost and the canonical boost, this initial rotation around the z-axis does not affect the momentum after the boost. To align with the convention of the helicity variables commonly used in the literature, we therefore adopt the rotation $R(\phi,\theta,-\phi)$. Analogously to Eqs. (18) and (19), the two-dimensional helicity and canonical boosts in the left-handed representation are

$$\begin{aligned}L_h(\mathbf{p})&=R(\phi,\theta,-\phi)B_z(\eta)\\&=\begin{pmatrix}ce^{\frac{\eta}{2}}&(-s^*)e^{-\frac{\eta}{2}}\\se^{\frac{\eta}{2}}&ce^{-\frac{\eta}{2}}\end{pmatrix},\end{aligned}\tag{312}$$

and

$$\begin{aligned}L_c(\mathbf{p})&=R(\phi,\theta,-\phi)B_z(\eta)R(\phi,\theta,-\phi)^{-1}\\&=\begin{pmatrix}c^2e^{\frac{\eta}{2}}+|s|^2e^{-\frac{\eta}{2}}&cs^*e^{\frac{\eta}{2}}-cs^*e^{-\frac{\eta}{2}}\\cse^{\frac{\eta}{2}}-cse^{-\frac{\eta}{2}}&|s|^2e^{\frac{\eta}{2}}+c^2e^{-\frac{\eta}{2}}\end{pmatrix}.\end{aligned}\tag{313}$$

Using the helicity boost and the canonical boost, the spinor-helicity variables and the canonical-spinor variables for a general momentum p can be obtained from Eq.

(309) and written as

$$\begin{aligned}\lambda_{\alpha I}(p)_h &= L_h(\mathbf{p})\lambda_{\alpha I}(k)\\ &= \sqrt{m}\begin{pmatrix} c\mathrm{e}^{\frac{\eta}{2}} & (-s^*)\mathrm{e}^{-\frac{\eta}{2}} \\ s\mathrm{e}^{\frac{\eta}{2}} & c\mathrm{e}^{-\frac{\eta}{2}}\end{pmatrix},\end{aligned} \tag{314}$$

and

$$\begin{aligned}\lambda_{\alpha I}(p)_c &= L_c(\mathbf{p})\lambda_{\alpha I}(k)\\ &= \sqrt{m}\begin{pmatrix} c^2\mathrm{e}^{\frac{\eta}{2}}+|s|^2\mathrm{e}^{-\frac{\eta}{2}} & cs^*\mathrm{e}^{\frac{\eta}{2}}-cs^*\mathrm{e}^{-\frac{\eta}{2}} \\ cs\mathrm{e}^{\frac{\eta}{2}}-cs\mathrm{e}^{-\frac{\eta}{2}} & |s|^2\mathrm{e}^{\frac{\eta}{2}}+c^2\mathrm{e}^{-\frac{\eta}{2}}\end{pmatrix}.\end{aligned} \tag{315}$$

Unless otherwise stated, the canonical-spinor variables will be used and denoted simply as $\lambda_{\alpha I}$ and $\tilde{\lambda}^I_{\dot\alpha}$. Conventionally, angle brackets $\langle\cdots\rangle$ and square brackets $[\cdots]$ are used to denote spinor contractions. The angle and square brackets correspond to left- and right-handed spinors and are defined by

$$\begin{aligned}
|p_I\rangle = \lambda_{\alpha I} &= \sqrt{m}\begin{pmatrix} c^2\mathrm{e}^{\frac{\eta}{2}}+|s|^2\mathrm{e}^{-\frac{\eta}{2}} & cs^*\mathrm{e}^{\frac{\eta}{2}}-cs^*\mathrm{e}^{-\frac{\eta}{2}} \\ cs\mathrm{e}^{\frac{\eta}{2}}-cs\mathrm{e}^{-\frac{\eta}{2}} & |s|^2\mathrm{e}^{\frac{\eta}{2}}+c^2\mathrm{e}^{-\frac{\eta}{2}}\end{pmatrix}_{\alpha I},\\
[p^I| = \tilde{\lambda}^I_{\dot\alpha} &= \sqrt{m}\begin{pmatrix} c^2\mathrm{e}^{\frac{\eta}{2}}+|s|^2\mathrm{e}^{-\frac{\eta}{2}} & cs^*\mathrm{e}^{\frac{\eta}{2}}-cs^*\mathrm{e}^{-\frac{\eta}{2}} \\ cs\mathrm{e}^{\frac{\eta}{2}}-cs\mathrm{e}^{-\frac{\eta}{2}} & |s|^2\mathrm{e}^{\frac{\eta}{2}}+c^2\mathrm{e}^{-\frac{\eta}{2}}\end{pmatrix}^I_{\dot\alpha},\\
\langle p_I| = \lambda^\alpha_I &= \sqrt{m}\begin{pmatrix} cs\mathrm{e}^{\frac{\eta}{2}}-cs\mathrm{e}^{-\frac{\eta}{2}} & -(c^2\mathrm{e}^{\frac{\eta}{2}}+|s|^2\mathrm{e}^{-\frac{\eta}{2}}) \\ |s|^2\mathrm{e}^{\frac{\eta}{2}}+c^2\mathrm{e}^{-\frac{\eta}{2}} & -(cs^*\mathrm{e}^{\frac{\eta}{2}}-cs^*\mathrm{e}^{-\frac{\eta}{2}})\end{pmatrix}^\alpha_I,\\
|p^I] = \tilde{\lambda}^{\dot\alpha I} &= \sqrt{m}\begin{pmatrix} cs^*\mathrm{e}^{\frac{\eta}{2}}-cs^*\mathrm{e}^{-\frac{\eta}{2}} & |s|^2\mathrm{e}^{\frac{\eta}{2}}+c^2\mathrm{e}^{-\frac{\eta}{2}} \\ -(c^2\mathrm{e}^{\frac{\eta}{2}}+|s|^2\mathrm{e}^{-\frac{\eta}{2}}) & -(cs\mathrm{e}^{\frac{\eta}{2}}-cs\mathrm{e}^{-\frac{\eta}{2}})\end{pmatrix}^{\dot\alpha I}.
\end{aligned} \tag{316}$$

Some basic spinor contractions can be written as:

$$\begin{aligned}\langle p_I q_J\rangle &= \lambda^\alpha_I(p)\lambda_{\alpha J}(q) = \varepsilon^{\alpha\beta}\lambda_{\beta I}(p)\lambda_{\alpha J}(q),\\ [p^I q^J] &= \tilde{\lambda}^I_{\dot\alpha}(p)\tilde{\lambda}^{\dot\alpha J}(q) = \varepsilon_{\dot\alpha\dot\beta}\tilde{\lambda}^{\dot\beta I}(p)\tilde{\lambda}^{\dot\alpha J}(q),\end{aligned} \tag{317}$$

$$\begin{aligned}\langle q_I|p|k^J] &= \lambda^\alpha_I(q)p_{\alpha\dot\alpha}\tilde{\lambda}^{\dot\alpha J}(k),\\ [q^I|p_1p_2|k^J] &= \tilde{\lambda}^I_{\dot\alpha}(q)p_1^{\dot\alpha\alpha}p_{2\alpha\dot\beta}\tilde{\lambda}^{\dot\beta J}(k).\end{aligned} \tag{318}$$

For the little group indices I,J, raising and lowering are performed with $\varepsilon_{IJ}/\varepsilon^{IJ}$, which allows conversion between upper and lower indices. More general contractions with multiple momenta inserted between spinors can be obtained following the above rules. In what follows, for notational simplicity, spinors will be labeled by the particle index, so that $\langle p_i| \to \langle i|$ denotes the spinor variables associated with the i-th particle.

The above summarizes the basic kinematics of particle scattering amplitudes. For an n-point scattering process, the amplitude is a function of the external momenta and the little group indices, (p_a,σ_a), $a=1,\cdots,n$. Under a Lorentz transformation Λ, it transforms with the momenta and the corresponding little group actions as

$$A_{\sigma_a}(p_a) = \prod_a D^{\sigma'_a(s_a)}_{\sigma_a}(W)A_{\sigma'_a}\Big((\Lambda p)_a\Big), \tag{319}$$

where $D^{\sigma'_a(s_a)}_{\sigma_a}(W)$ is the spin-s_a representation matrix of the little group rotation associated with particle-a. Motivated by this, momenta are decomposed into spinor variables carrying both Lorentz and little group indices. By contracting the Lorentz indices of these spinor variables, amplitudes can be constructed that are Lorentz invariant and transform under the little group. The resulting amplitudes depend only on these spinor variables and, by construction, satisfy Eq. (319).

For example, in the case $n=3$, consider the three-point amplitude $3\to 1+2$, where each particle is massive and has spins $s_3=\dfrac{1}{2}, s_1=\dfrac{1}{2}, s_2=1$. The amplitude constructed from spinor variables is

$$\begin{aligned}\mathcal{A}^{\{J_1\},\{I_1I_2\}}_{\{K_1\}} &= \kappa_1\langle 1^{J_1}2^{(I_1}\rangle[2^{I_2)}3_{K_1}] + \kappa_2[1^{J_1}2^{(I_1}]\langle 2^{I_2)}3_{K_1}\rangle\\ &\quad+\kappa_3\langle 1^{J_1}2^{(I_1}\rangle\langle 2^{I_2)}3_{K_1}\rangle + \kappa_4[1^{J_1}2^{(I_1}][2^{I_2)}3_{K_1}],\end{aligned} \tag{320}$$

where (I_1I_2) denotes the symmetrization of the indices, and $\kappa_{1,2,3,4}$ are coefficients determined by experiment. It can be seen that the amplitude constructed in this way is Lorentz invariant and satisfies the required little group transformation properties,

$$\begin{aligned}\mathcal{A}^{\{J_1\},\{I_1I_2\}}_{\{K_1\}}(p_3,p_1,p_2) &= (W_3)^{K'_1}_{K_1}(W_1^{-1})^{J_1}_{J'_1}(W_2^{-1})^{I_1}_{I'_1}(W_2^{-1})^{I_2}_{I'_2}\\ &\quad\times\mathcal{A}^{\{J'_1\},\{I'_1I'_2\}}_{\{K'_1\}}(\Lambda p_3,\Lambda p_1,\Lambda p_2).\end{aligned} \tag{321}$$

Moreover, this three-point amplitude has four LS partial waves: $(L,S)=(0,\dfrac{1}{2}),(1,\dfrac{1}{2}),(1,\dfrac{3}{2}),(2,\dfrac{3}{2})$. In total, there are four independent structures for the amplitude. Consequently, the four independent bases constructed from spinor contractions form a complete set.

These four spinor contractions represent the basic building blocks of the three-point amplitude with spins $s_3=\dfrac{1}{2}, s_1=\dfrac{1}{2}, s_2=1$.

To make contact with the conventional description based on Feynman rules, consider the most general vertex between two fermions and one vector: $\mathcal{L}_{\rm int}=\bar\psi_1\Gamma^\mu\psi_3V_\mu$, whose coupling structure Γ^μ can be written as

$$\Gamma_\mu = \gamma_\mu(g_L P_L + g_R P_R) + \mathrm{i}\sigma_{\mu\nu} p_2^\nu \big(f_L P_L + f_R P_R\big), \tag{322}$$

where $P_{L,R} = \dfrac{1}{2}(1 \mp \gamma_5)$, $\sigma_{\mu\nu} = \dfrac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$, and p_2 denotes the momentum carried by the vector field V^μ. The gamma matrices in our convention are given by

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_{\mu\alpha\dot\alpha} \\ \bar\sigma_\mu^{\dot\alpha\alpha} & 0 \end{pmatrix}, \gamma_5 = \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{323}$$

where $\sigma_{\mu\alpha\dot\alpha} = (\delta_{\alpha\dot\alpha}, \sigma_{\alpha\dot\alpha})$ and $\bar\sigma_\mu^{\dot\alpha\alpha} = (\delta^{\dot\alpha\alpha}, -\sigma^{\dot\alpha\alpha})$. The projection operators $P_{L,R}$ can be written as

$$P_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{324}$$

The decay amplitude for the process $\psi_3 \to \psi_1 V_2$ can be written as

$$\mathcal{A} = \epsilon^{*\mu}\bar u(p_1)\big(\gamma_\mu(g_L P_L + g_R P_R) + \mathrm{i}\sigma_{\mu\nu}p_2^\nu(f_L P_L + f_R P_R)\big)u(p_3), \tag{325}$$

where the four-component spinors are

$$u(p) = \begin{pmatrix} |p_K\rangle \\ |p_K] \end{pmatrix}, \quad \bar u(p) = \begin{pmatrix} -\langle p^J| & [p^J| \end{pmatrix}, \tag{326}$$

and the polarization vectors in bispinor form are:

$$\epsilon^*_{\alpha\dot\alpha} = \epsilon^{*\mu}\sigma_{\mu\alpha\dot\alpha} = -\frac{|p^{(I_1}\rangle[p^{I_2)}|}{m}, \quad \epsilon^{*\dot\alpha\alpha} = \epsilon^{*\mu}\bar\sigma_\mu^{\dot\alpha\alpha} = -\frac{|p^{(I_1}]\langle p^{I_2)}|}{m}, \tag{327}$$

where $(I_1 I_2)$ denotes the symmetrization of the indices. The amplitude, written in spinor variable form, is

$$\begin{aligned}\mathcal{A}^{\{J_1\},\{I_1 I_2\}}_{\{K_1\}} =& -g_R\langle 1^{J_1}|\epsilon^*_{\alpha\dot\alpha}|3_{K_1}] + g_L[1^{J_1}|\epsilon^{*\dot\alpha\alpha}|3_{K_1}\rangle \\ & - \frac{f_R}{2}[1^{J_1}|\epsilon^{*\dot\alpha\alpha}p_{2\alpha\dot\beta}|3_{K_1}] + \frac{f_L}{2}\langle 1^{J_1}|\epsilon^*_{\alpha\dot\alpha}p^{2\dot\alpha\beta}|3_{K_1}\rangle \\ =& \frac{g_R\langle 1^{J_1}2^{(I_1}\rangle[2^{I_2)}3_{K_1}] - g_L[1^{J_1}2^{(I_1}]\langle 2^{I_2)}3_{K_1}\rangle}{m_2} \\ & + \frac{f_R}{2}[1^{J_1}2^{(I_1}][2^{I_2)}3_{K_1}] - \frac{f_L}{2}\langle 1^{J_1}2^{(I_1}\rangle\langle 2^{I_2)}3_{K_1}\rangle.\end{aligned} \tag{328}$$

Taking $\kappa_1 = \dfrac{g_R}{m_2}, \kappa_2 = -\dfrac{g_L}{m_2}, \kappa_3 = -\dfrac{f_L}{2}$, and $\kappa_4 = \dfrac{f_R}{2}$ in Eq. (320), the corresponding expression is obtained.

More generally, three-point amplitudes for massive particles with arbitrary spins are discussed in Ref. [80]. Now consider the more general two-body decay amplitude $3 \to 1+2$, where the amplitude is a function of the canonical-spinor variables. The particle spins are s_1, s_2, and s_3, respectively. The amplitude can be written as

$$\mathcal{A}^{\{I_1^{(1)}\cdots I_{2s_1}^{(1)}\},\{I_1^{(2)}\cdots I_{2s_2}^{(2)}\}}_{\{I_1^{(3)}\cdots I_{2s_3}^{(3)}\}}\big((\lambda_{\alpha I}, \tilde\lambda^I_{\dot\alpha})_1; (\lambda_{\alpha I}, \tilde\lambda^I_{\dot\alpha})_2; (\lambda_{\alpha I}, \tilde\lambda^I_{\dot\alpha})_3\big). \tag{329}$$

Under a Lorentz transformation, the amplitude transforms as follows,

$$\begin{aligned}&\mathcal{A}^{\{I_1^{(1)}\cdots I_{2s_1}^{(1)}\},\{I_1^{(2)}\cdots I_{2s_2}^{(2)}\}}_{\{I_1^{(3)}\cdots I_{2s_3}^{(3)}\}}(p_i) \\ &\xrightarrow{\Lambda} \left(W^{K_1^{(3)}}_{I_1^{(3)}}\cdots W^{K_{2s_3}^{(3)}}_{I_{2s_3}^{(3)}}\right)\left((W^{-1})^{K_1^{(1)}}_{I_1^{(1)}}\cdots (W^{-1})^{I_{2s_1}^{(1)}}_{K_{2s_1}^{(1)}}\right) \\ &\times \left((W^{-1})^{I_1^{(2)}}_{K_1^{(2)}}\cdots (W^{-1})^{I_{2s_2}^{(2)}}_{K_{2s_2}^{(2)}}\right)\mathcal{A}^{\{K_1^{(1)}\cdots K_{2s_1}^{(1)}\},\{K_1^{(2)}\cdots K_{2s_2}^{(2)}\}}_{\{K_1^{(3)}\cdots K_{2s_3}^{(3)}\}}(\Lambda p_i).\end{aligned} \tag{330}$$

As discussed in Section II, amplitudes can be decomposed into LS partial waves. Accordingly, the amplitude constructed from canonical-spinor variables can be decomposed as

$$\mathcal{A}^{\{I_1^{(1)}\cdots I_{2s_1}^{(1)}\},\{I_1^{(2)}\cdots I_{2s_2}^{(2)}\}}_{\{I_1^{(3)}\cdots I_{2s_3}^{(3)}\}} = \sum_{LS} N_{LS}\mathcal{A}^{\{I_1^{(1)}\cdots I_{2s_1}^{(1)}\},\{I_1^{(2)}\cdots I_{2s_2}^{(2)}\}}_{\{I_1^{(3)}\cdots I_{2s_3}^{(3)}\}}(L,S), \tag{331}$$

where N_{LS} is the coefficient related to energy and mass, which can be measured from experiments. Here, $\mathcal{A}(L,S)$ denotes the canonical-spinor amplitude in the LS expansion.

The decay amplitude in any frame can be expressed as the amplitude in the COM frame, transformed by a little group rotation. The COM frame amplitude itself can be decomposed into orbital (L) and spin (S) parts. Consequently, the amplitude in any frame can be separated into covariant L and S components via a spinor decomposition. The corresponding LS amplitude is defined as

$$\mathcal{A}^{\{I_1^{(1)}\cdots I_{2s_1}^{(1)}\},\{I_1^{(2)}\cdots I_{2s_2}^{(2)}\}}_{\{I_1^{(3)}\cdots I_{2s_3}^{(3)}\}}(L,S) \equiv \mathcal{S}^{\{I_1^{(1)}\cdots I_{2s_1}^{(1)}\},\{I_1^{(2)}\cdots I_{2s_2}^{(2)}\}}_{\{K\}}\mathcal{Y}^L_{\{I\}}C^{S,\{K\};L,\{I\}}_{s_3,\{I_1^{(3)}\cdots I_{2s_3}^{(3)}\}}. \tag{332}$$

Here

- $\mathcal{Y}^L_{\{I\}}$ (defined in Eq. (337)) encodes the orbital angular momentum L structure. For L (orbital angular momentum quantum number), the function $\mathcal{Y}^L_{\{I\}}$ carries $2L$ symmetric little group $SU(2)$ indices $\{I\} = (I_1\cdots I_{2L})$.

- $\mathcal{S}^{\{I_1^{(1)}\cdots I_{2s_1}^{(1)}\},\{I_1^{(2)}\cdots I_{2s_2}^{(2)}\}}_{\{K\}}$ (defined in Eq. (340)), which couples the spins s_1 and s_2 into a total spin-S, not only carries $2s_1$ and $2s_2$ symmetric little group indices $\{I_1^{(1)}\cdots I_{2s_1}^{(1)}\},\{I_1^{(2)}\cdots I_{2s_2}^{(2)}\}$ associated with the individual spins, but also includes $2S$ symmetric little group indices $\{K\}=(K_1\cdots K_{2S})$ corresponding to the total spin-S representation resulting from the coupling.

By coupling the indices $\{I\}$ carried by $\mathcal{Y}^L_{\{I\}}$ with the indices $\{K\}$ carried by $\mathcal{S}^{\{I^{(1)}_1\cdots I^{(1)}_{2s_1}\},\{I^{(2)}_1\cdots I^{(2)}_{2s_2}\}}_{\{K\}}$ via the $SU(2)$ CGCs, the final amplitude $\mathcal{A}(L,S)$ is obtained in Eq. (332).

For the canonical-spinor amplitude, we can choose to normalize the amplitude $\mathcal{A}^{\{I^{(1)}\}\{I^{(2)}\}}_{\{I^{(3)}\}}(L,S)$, which allows for the separate extraction of normalization factors for the orbital part $\mathcal{Y}^L_{\{I\}}$ and the spin part $\mathcal{S}^{\{I^{(1)}\},\{I^{(2)}\}}_{\{K\}}$. The normalized amplitudes satisfy

$$\int d\Phi\left(\mathcal{A}^{\{I^{(1)}\}\{I^{(2)}\}}_{\{I^{(3)}\}}(L,S)\right)^*\mathcal{A}^{\{I^{(1)}\}\{I^{(2)}\}}_{\{I^{(3)}\}}(L',S')=\delta^{S'}_S\delta^{L'}_L. \quad (333)$$

Before constructing the orbital and spin parts of the amplitude, the $SU(2)$ CGCs are first defined as

$$C^{s_1,\{I\};s_2,\{K\}}_{s,\{J\}}\equiv\sqrt{\frac{(a+b)!(b+c)!(a+c+1)!}{b!(a+b+c+1)!a!c!}}\times\delta^{(I_1\cdots I_a}_{(J_1\cdots J_a}\varepsilon^{I_{a+1}\cdots I_{a+b}),(K_1\cdots K_b}\delta^{K_{b+1}\cdots K_{b+c})}_{J_{a+1}\cdots J_{a+c})}, \quad (334)$$

with

$$a=s_1+s-s_2,\quad b=s_1+s_2-s,\quad c=s_2+s-s_1,$$
$$\delta^{I_1\cdots I_a}_{J_1\cdots J_a}=\delta^{I_1}_{J_1}\cdots\delta^{I_a}_{J_a},\quad \varepsilon^{I_1\cdots I_b,K_1\cdots K_b}=\varepsilon^{I_1K_1}\cdots\varepsilon^{I_bK_b}, \quad (335)$$

where $(J_1\cdots J_{a+c})$, $(I_1\cdots I_{a+b})$, and $(K_1\cdots K_{b+c})$ denote symmetrization over the corresponding indices. In subsection C.2, there are some examples of calculating the $SU(2)$ CGC.

For the orbital part $\mathcal{Y}^L_{\{I\}}$, it is convenient to begin in the COM frame. In this frame, the orbital structure depends explicitly on the momenta of the final particles. Momentum conservation ensures that the momenta of the two final particles satisfy $\mathbf{p}^*_1=-\mathbf{p}^*_2$. Therefore, the relative motion is fully characterized by the momentum of a single particle; here, particle-1 is chosen. The most elementary building block of the orbital part can be constructed from the momentum of particle-1, which can be written as

$$\langle 3_{I_1}|p_1|3_{I_2}]. \quad (336)$$

Owing to the Lorentz covariance of the canonical-spinor amplitude, such a building block, once properly constructed, can be promoted to any frame while still encoding the correct orbital angular momentum information. As a result, the orbital structure $\mathcal{Y}^L_{\{I\}}$ in any frame can be defined as

$$\mathcal{Y}^L_{\{I\}}\equiv\sqrt{\frac{(2(2L+1)!)}{\pi|2\mathbf{p}_{1\mathrm{com}}|^{2L+1}m_3^{2L-1}L!^2}}(\langle 3|p_1|3]^L)_{\{I\}}, \quad (337)$$

with

$$(\langle 3|p_1|3]^L)_{\{I\}}=\langle 3_{(I_1}|p_1|3_{I_2}]\cdots\langle 3_{I_{2L-1}}|p_1|3_{I_{2L})}], \quad (338)$$

where L is the orbital angular momentum, $|\mathbf{p}_{1\mathrm{com}}|$ is the magnitude of the relative momentum in the COM frame, m_3 is the mass of the parent particle (particle-3), and $(I_1\cdots I_{2L})$ denotes symmetrization over the indices. The normalization factor in front of $(\langle 3|p_1|3]^L)_{\{I\}}$ arises from the normalization condition in Eq. (333).

For the spin part $\mathcal{S}^{\{I^{(1)}\},\{I^{(2)}\}}_{\{K\}}$, it is convenient to begin in the COM frame. In this frame, appropriate spinor contractions are constructed whose structure matches the required number of little group indices. Additionally, these contractions must reduce to a form proportional to the identity matrix in the COM frame, since no further little group transformation is needed there. To this end, a set of spinor structures, denoted by τ, is identified that satisfies these criteria. The explicit form of τ is

$$\tau^I_J(p_3,p_1)\equiv\langle 1^I3_J\rangle-[1^I3_J]\overset{\mathrm{COM}}{=}\sqrt{2m_3(E_{1\mathrm{com}}+m_1)}\begin{pmatrix}1&0\\0&1\end{pmatrix}. \quad (339)$$

In the COM frame, this tensor is proportional to the identity matrix. In any frame, its form remains covariant, and it behaves as an effective little group transformation acting on the spin indices.

With this building block, the spin part $\mathcal{S}^{\{I^{(1)}\},\{I^{(2)}\}}_{\{K\}}$ can be defined by coupling the τ tensors for particle-1 and particle-2 through CGCs. The spin part of the amplitude is defined as

$$\mathcal{S}^{\{I^{(1)}_1\cdots I^{(1)}_{2s_1}\},\{I^{(2)}_1\cdots I^{(2)}_{2s_2}\}}_{\{K\}}\equiv\frac{1}{(m_3(E_{1\mathrm{com}}+m_1))^{s_1}(m_3(E_{2\mathrm{com}}+m_2))^{s_2}}(\tau_1^{2s_1})^{\{I^{(1)}\}}_{\{J^{(1)}\}}(\tau_2^{2s_2})^{\{I^{(2)}\}}_{\{J^{(2)}\}}\times C^{s_1,\{J^{(1)}\};s_2,\{J^{(2)}\}}_{S,\{K\}}, \quad (340)$$

with

$$(\tau_1^{2s_1})^{\{I^{(1)}\}}_{\{J^{(1)}\}}=\tau^{(I^{(1)}_1}_{(J^{(1)}_1}\tau^{I^{(1)}_2}_{J^{(1)}_2}\cdots\tau^{I^{(1)}_{2s_1})}_{J^{(1)}_{2s_1})}(p_3,p_1),$$
$$(\tau_2^{2s_2})^{\{I^{(2)}\}}_{\{J^{(2)}\}}=\tau^{(I^{(2)}_1}_{(J^{(2)}_1}\tau^{I^{(2)}_2}_{J^{(2)}_2}\cdots\tau^{I^{(2)}_{2s_2})}_{J^{(2)}_{2s_2})}(p_3,p_2), \quad (341)$$

where $(J_1^{(1)}\cdots J_{2s_1}^{(1)})$, $(I_1^{(1)}\cdots I_{2s_1}^{(1)})$, $(J_1^{(2)}\cdots J_{2s_2}^{(2)})$, and $(I_1^{(2)}\cdots I_{2s_2}^{(2)})$ denote symmetrization over the corresponding indices. The normalization factor in front of τ arises from the normalization condition in Eq. (333).

After constructing the orbital part $\mathcal{Y}^L_{\{I\}}$ and the spin part $\mathcal{S}^{\{I^{(1)}\},\{I^{(2)}\}}_{\{K\}}$, these can be combined to obtain the canonical-spinor amplitude in the LS expansion, given in Eq. (332).

The canonical-spinor amplitude and the canonical amplitude are represented in two different bases. The canonical-spinor amplitude is defined in the canonical-spinor basis $|s,\{I\}\rangle$, while the canonical amplitude is defined in the canonical basis $|s,\sigma\rangle$. Between the two bases, there exists a corresponding transformation, which can be written as

$$|s,\sigma\rangle = \sqrt{\frac{(2s)!}{(s+\sigma)!(s-\sigma)!}}|s,\{I\}\rangle, \tag{342}$$

where the indices $\{I\}$ denote the symmetric combination $(I_1\cdots I_{2s})$, with $I_k=\pm\frac{1}{2}$ for $k=1,\cdots,2s$, and represent any choice of $\{I\}$ such that

$$\sigma=\sum_{k=1}^{2s}I_k. \tag{343}$$

Therefore, the relation between the canonical amplitude in Eq. (73) and the canonical-spinor amplitude in Eq. (332) can be expressed as

$$\begin{aligned}\mathcal{A}^{\sigma_1\sigma_2}_{\sigma_3}(L,S) = N&\sqrt{\frac{(2s_3)!}{(s_3+\sigma_3)!(s_3-\sigma_3)!}}\\ &\times\sqrt{\frac{(2s_1)!}{(s_1+\sigma_1)!(s_1-\sigma_1)!}}\sqrt{\frac{(2s_2)!}{(s_2+\sigma_2)!(s_2-\sigma_2)!}}\\ &\times\mathcal{A}^{\{I_1\},\{I_2\}}_{\{I_3\}}(p_3,p_1,p_2,L,S),\end{aligned} \tag{344}$$

where N is the coefficient related to energy and mass. A detailed discussion on how to determine N in the different bases is given in Appendix D. The factors in front of the spin part $\mathcal{S}^{\{I^{(1)}\},\{I^{(2)}\}}_{\{K\}}$ and the orbital part $\mathcal{Y}^L_{\{I\}}$ in Eq. (332) can be chosen such that $N=1$. For this new choice, the spin and orbital parts will be denoted by $\mathcal{S}'^{\{I^{(1)}\},\{I^{(2)}\}}_{\{K\}}$ and $\mathcal{Y}'^L_{\{I\}}$, respectively.

The spin part $\mathcal{S}'^{\{I^{(1)}\},\{I^{(2)}\}}_{\{K\}}$ is defined as

$$\mathcal{S}'^{\{I^{(1)}\},\{I^{(2)}\}}_{\{K\}} \equiv (\tau_1'^{2s_1})^{\{I^{(1)}\}}_{\{J^{(1)}\}}(\tau_2'^{2s_2})^{\{I^{(2)}\}}_{\{J^{(2)}\}}C^{s_1,\{J^{(1)}\};s_2,\{J^{(2)}\}}_{S,\{K\}}, \tag{345}$$

where τ' is defined as

$$\begin{aligned}\tau'^{I^{(1)}}_{J^{(1)}}(p_3,p_1) &\equiv \frac{1}{\sqrt{2m_3(E_{1\text{com}}+m_1)}}(\langle 1^{I^{(1)}}3_{J^{(1)}}\rangle - [1^{I^{(1)}}3_{J^{(1)}}]),\\ \tau'^{I^{(2)}}_{J^{(2)}}(p_3,p_2) &\equiv \frac{1}{\sqrt{2m_3(E_{2\text{com}}+m_2)}}(\langle 2^{I^{(2)}}3_{J^{(2)}}\rangle - [2^{I^{(2)}}3_{J^{(2)}}]).\end{aligned} \tag{346}$$

The orbital part $\mathcal{Y}'^L_{\{I\}}$ is defined as

$$\mathcal{Y}'^L_{\{I\}} \equiv \frac{(-1)^L}{(m_3|\mathbf{p}_{1\text{com}}|)^L}\sqrt{\frac{(2L+1)}{4\pi a_L}}\langle 3|p_1|3]^L. \tag{347}$$

with

$$a_L=\prod_{k=1}^{L}\frac{2k}{2k-1},\quad a_0=1. \tag{348}$$

Using Eq. (342) to convert the indices of the orbital part $\mathcal{Y}'^L_{\{I\}}$, the spherical harmonic is obtained and can be written as

$$Y^L_{\sigma_L}=\sqrt{\frac{(2L)!}{(L+\sigma_L)!(L-\sigma_L)!}}\mathcal{Y}'^L_{\{I\}}. \tag{349}$$

Under this choice, the factor N reduces to 1.

This allows a direct comparison with the traditional-LS amplitude in Eq. (73), which can be written as

$$A^{\sigma_1\sigma_2}_{\sigma_3}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2;L,S) = \underbrace{D^{\sigma_1'(s_1)*}_{\sigma_1}(R_{31})\,D^{\sigma_2'(s_2)*}_{\sigma_2}(R_{32})}_{\text{little group}}\underbrace{C^{s_1,\sigma_1';s_2,\sigma_2'}_{S,\sigma_S}}_{\text{The CGC of } s_1s_2\to S}\times\underbrace{C^{L,\sigma_L;S,\sigma_S}_{s_3,\sigma_3}}_{\text{The CGC of } LS\to s_3}\underbrace{Y^L_{\sigma_L}(\Omega)}_{\text{spherical harmonic}}, \tag{350}$$

$$\mathcal{A}'^{\{I^{(1)}\},\{I^{(2)}\}}_{\{I^{(3)}\}}(p_3,p_1,p_2;L,S) = \underbrace{(\tau_1'^{2s_1})^{\{I^{(1)}\}}_{\{J^{(1)}\}}(\tau_2'^{2s_2})^{\{I^{(2)}\}}_{\{J^{(2)}\}}}_{\text{little group}}\underbrace{C^{s_1,\{J^{(1)}\};s_2,\{J^{(2)}\}}_{S,\{K\}}}_{\text{The CGC of } s_1s_2\to S}\times\underbrace{C^{S,\{K\};L,\{I\}}_{s_3,\{I^{(3)}\}}}_{\text{The CGC of } LS\to s_3}\underbrace{\mathcal{Y}'^L_{\{I\}}}_{\text{spherical harmonic}}. \tag{351}$$

By comparing the two LS partial wave amplitudes in any frame, one finds that the traditional-LS amplitude is not manifestly Lorentz covariant. It requires boosting all momenta to the two-body COM frame, evaluating the amplitude once in the COM frame, and then using little group rotations to transform the amplitude to any frame. By contrast, the canonical-spinor amplitude is defined in an arbitrary reference frame. It does not require boosting the event back to the COM frame, and the amplitude can be evaluated directly from the momenta in the chosen frame.

Moreover, when returning to the COM frame, τ in the canonical-spinor construction becomes the identity 1, and the spin part no longer contains momentum information, so it is fully equivalent to the traditional-LS partial waves. Meanwhile, in any frame, the covariant structure τ'^{I}_{J} ensures that the spin coupling is carried out within the same little group representation, and therefore the final result is consistent with the traditional-LS amplitude.

B. Comparison with the covariant projection tensor method

In subsection III.B, the covariant projection tensor method is presented in both the GS scheme and the PS scheme. Although both schemes lead to expressions that can be evaluated in any frame, neither scheme simultaneously achieves an explicit LS separation and a manifestly covariant definition. Therefore, we rewrite the covariant projection tensor method in spinor form to make the covariance and the LS separation explicit.

In Ref. [48], angular momentum coupling in the Lorentz representation is organized using irreducible representations (s_L,s_R) of $SO(3,1)$, where s_L and s_R are integers or half-integers. For an IRREP (s_L,s_R), the complex conjugate representation is (s_R,s_L). Therefore, for a representation $[\ell]$ the following convention is adopted,

$$[\ell] \equiv \begin{cases} (s_L,s_R) & \text{for } s_L = s_R, \\ (s_L,s_R)\oplus(s_R,s_L) & \text{for } s_L \neq s_R. \end{cases} \tag{352}$$

For the spin wave function at the standard momentum $k^\mu = (m,\mathbf{0})$, consider a massive particle with spin-s. Its associated IRREP $[\ell] = (s_L,s_R) = [l]\otimes[r]$ is defined as

$$\begin{aligned} u^{\ell}_{\sigma}(k;s) &\equiv u^{\ell}_{\sigma}(k;[\ell],s) = u^{lr}_{\sigma}(k;(s_L,s_R),s), \\ \bar{u}^{\sigma}_{\ell}(k;s) &\equiv \bar{u}^{\sigma}_{\ell}(k;[\ell],s) = \bar{u}^{\sigma}_{lr}(k;(s_L,s_R),s), \end{aligned} \tag{353}$$

where $l=-s_L,-s_L+1,\cdots,s_L$ and $r=-s_R,-s_R+1,\cdots,s_R$ correspond to the IRREP indices of $SU(2)_L$ and $SU(2)_R$, respectively. Note that, in our convention, the placement of the indices on the wave functions u and $\bar{u}$ differs from Ref. [48].

The spin wave functions $u^{lr}_{\sigma}(k;(s_L,s_R),s)$ and $\bar{u}^{\sigma}_{lr}(k;(s_L,s_R),s)$ are essentially the CGCs[1)], which can be written as

$$u^{lr}_{\sigma}(k;(s_L,s_R),s) = C^{s_L,\sigma_l;s_R,\sigma_r}_{s,\sigma}, \quad \bar{u}^{\sigma}_{lr}(k;(s_L,s_R),s) = C^{s,\sigma}_{s_L,\sigma_l;s_R,\sigma_r}. \tag{354}$$

The spin wave functions satisfy the orthonormality relation:

$$\bar{u}^{\sigma_1}_{\ell}(k;s)u^{\ell}_{\sigma_2}(k;s) = \delta^{\sigma_2}_{\sigma_1}. \tag{355}$$

From Eq. (192), the spin wave function in any frame can be written as

$$\begin{aligned} u^{lr}_{\sigma}(p;(s_L,s_R),s) &= D^{l}{}_{l'}(L(\mathbf{p}))D^{r}{}_{r'}(L(\mathbf{p}))u^{l'r'}_{\sigma}(k;(s_L,s_R),s), \\ \bar{u}^{\sigma}_{lr}(p;(s_L,s_R),s) &= D^{l'}{}_{l}(L^{-1}(\mathbf{p}))D^{r'}{}_{r}(L^{-1}(\mathbf{p}))\bar{u}^{\sigma}_{l'r'}(k;(s_L,s_R),s). \end{aligned} \tag{356}$$

From Eqs. (309) and (316), it follows that the spin wave functions in the left- and right-handed representations can be constructed from spinors. For spin-$1/2$, the spin wave functions can be written as

$$u^{l0}_{\sigma}(p;(\frac{1}{2},0),\frac{1}{2}) \to U^{l}{}_{\alpha}\lambda^{\alpha}_{I}, \qquad u^{0r}_{\sigma}(p;(0,\frac{1}{2}),\frac{1}{2}) \to \tilde{\lambda}^{\dot{\alpha}}_{I}, \tag{357}$$

$$\bar{u}^{\sigma}_{0r}(p;(\frac{1}{2},0),\frac{1}{2}) \to \tilde{\lambda}^{I}_{\dot{\alpha}}, \;\; \bar{u}^{\sigma}_{l0}(p;(0,\frac{1}{2}),\frac{1}{2}) \to -(U^{-1})^{\alpha}_{l}\lambda^{I}_{\alpha}, \tag{358}$$

where $U^{l}{}_{\alpha}$ is a constant matrix that implements the change of notation between the indices (l,r) and the spinor indices $(\alpha,\dot{\alpha})$. In the (l,r) notation, the four-dimensional Pauli matrices are $(\sigma_\mu)^{l}{}_{r}$, whereas in the spinor notation they are denoted by $(\sigma_\mu)_{\alpha\dot{\alpha}}$. With this convention, one chooses

$$U^{l}{}_{\alpha} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad (U^{-1})^{\alpha}_{l} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{359}$$

More generally, the spin wave functions with arbitrary spin-s can be constructed from basic spinors and can be written as

1) A more detailed derivation can be found in Section VII of Chapter V in Ref. [84].

$$\bar{u}_\ell^{\{I\}}(p;s) \equiv \left(|p\rangle^{2s_L}[p|^{2s_R}\right)_\ell^{\{I\}}, \quad u_{\{I\}}^\ell(p;s) \equiv \left(|p]^{2s_R}\langle p|^{2s_L}\right)_{\{I\}}^\ell, \tag{360}$$

where $[s_L, s_R]$ denotes the irreducible representation of the Lorentz group. For simplicity, the constant matrices U that relate the (l,r) indices to the spinor indices $(\alpha,\dot{\alpha})$, as well as the normalization factors of the spin wave functions, are omitted here.

Starting from the definition of the spin wave functions, all the spin wave functions are written as functions of the parent momentum and the spin quantum numbers of the individual particles, as shown in Eq. (360). The rewritten amplitude is:

$$\begin{aligned}&\mathcal{A}_{\{I_3\}}^{\{I_1\}\{I_2\}}(p_3,p_1,p_2;L,S)\\ &\equiv \Gamma_{\ell_3}^{\ell_1\ell_2}(p_3,p_1,p_2;L,S)(\tau_1^{2s_1})_{\{J\}}^{\{I_1\}}(\tau_2^{2s_2})_{\{K\}}^{\{I_2\}}\\ &\quad\times u_{\{I_3\}}^{\ell_3}(p_3;s_3)\bar{u}_{\ell_1}^{\{J\}}(p_3;s_1)\bar{u}_{\ell_2}^{\{K\}}(p_3;s_2),\end{aligned} \tag{361}$$

where the factor $\tau_{\{J\}}^{\{I\}}$ can be obtained from Eq. (341). The role of $\tau_{\{J\}}^{\{I\}}$ is analogous to the little group rotation part in Eq. (250). For the coupling structure $\Gamma_{\ell_3}^{\ell_1\ell_2}$, its definition is the same as in Eq. (223), and the covariant projection tensor can be written as

$$\begin{aligned}&P_{\ell_3}^{\ell_1\ell_2}(p_3;s_3,s_1,s_2)\\ &= \bar{u}_{\ell_3}^{\{I_3\}}(p_3,s_3)u_{\{I_1\}}^{\ell_1}(p_3,s_1)u_{\{I_2\}}^{\ell_2}(p_3,s_2)C_{s_3,\{I_3\}}^{s_1,\{I_1\};s_2,\{I_2\}}\\ &= (|3\rangle^{2s_{3L}}[3|^{2s_{3R}})_{\ell_3}^{\{I_3\}}\\ &\quad\times(|3]^{2s_{1R}}\langle 3|^{2s_{1L}})_{\{I_1\}}^{\ell_1}(|3]^{2s_{2R}}\langle 3|^{2s_{2L}})_{\{I_2\}}^{\ell_2}C_{s_3,\{I_3\}}^{s_1,\{I_1\};s_2,\{I_2\}},\end{aligned} \tag{362}$$

where the pair $[s_{iL}, s_{iR}]$ $(i=1,2,3)$ represents the Lorentz group irreducible representation of the $[\ell_i]$.

For the orbital part $t_{\ell_L}^L$, one has

$$t_{\ell_L}^L(p_3;p_1) = \mathcal{Y}_{\{I\}}^L(p_3,p_1)\times(|3\rangle^L[3|^L)_{\ell_L}^{\{I\}}, \tag{363}$$

where $\mathcal{Y}_{\{I\}}^L$ has already been defined in Eq. (337).

Having established the explicit structures, the covariant projection tensor amplitude in Eq. (250) and the canonical-spinor amplitude in Eq. (361) can now be compared. In any frame, the two amplitudes can be written as

$$\begin{aligned}A_{\sigma_3}^{\sigma_1\sigma_2}(p_3,p_1,p_2;L,S) &= \Gamma_{\ell_3}^{\ell_1\ell_2}(k_3,p_1^*,p_2^*;L,S)u_{\sigma_3}^{\ell_3}(k_3;s_3)\times\underbrace{D_{\ell_1}^{\ell_1'}(R_{31}^{-1})\bar{u}_{\ell_1'}^{\sigma_1}(k_1;s_1)}_{\text{LG + wave function}}\,\underbrace{D_{\ell_2}^{\ell_2'}(R_{32}^{-1})\bar{u}_{\ell_2'}^{\sigma_2}(k_2;s_2)}_{\text{LG + wave function}},\\ \mathcal{A}_{\{I_3\}}^{\{I_1\}\{I_2\}}(p_3,p_1,p_2;L,S) &= \Gamma_{\ell_3}^{\ell_1\ell_2}(p_3,p_1,p_2;L,S)u_{\{I_3\}}^{\ell_3}(p_3;s_3)\times\underbrace{\tau_{1\{J\}}^{\{I_1\}}\bar{u}_{\ell_1}^{\{J\}}(p_3;s_1)}_{\text{LG + wave function}}\,\underbrace{\tau_{2\{K\}}^{\{I_2\}}\bar{u}_{\ell_2}^{\{K\}}(p_3;s_2)}_{\text{LG + wave function}}.\end{aligned} \tag{364}$$

By comparison, the canonical-spinor-based covariant projection tensor amplitude preserves explicit covariance and does not require boosting the event back to the two-body COM frame. Moreover, the covariant projection tensor amplitude, rewritten in terms of the canonical-spinor variables, is equivalent to the canonical-spinor amplitude. This equivalence originates from the full contraction of all Lorentz indices in the covariant projection tensor amplitude, rewritten with canonical-spinor variables, and the result of this contraction is exactly the canonical-spinor amplitude. This also ensures that, when returned to the COM frame, the orbital and spin parts are completely separated, maintaining consistency with the traditional-LS amplitude.

C. Examples of canonical-spinor amplitude calculation

In this subsection, several explicit examples of canonical-spinor amplitude calculations are presented. Using Eqs. (316) and (317), the following basic spinor structures are obtained:

$$p_{\alpha\dot{\alpha}} = |p_{\alpha I}\rangle[p_{\dot{\alpha}}^I|, \quad p_{\dot{\alpha}\alpha} = |p^{\dot{\alpha}I}]\langle p_I^\alpha|, \tag{365}$$

$$\langle p_I q^J\rangle = -\langle q^J p_I\rangle, \quad [p_J q^I] = -[q^I p_J], \tag{366}$$

$$\begin{aligned}&\langle p^I p_J\rangle = m\delta_J^I, \quad [p_J p^I] = m\delta_J^I,\\ &|p_I\rangle\langle p^I| = m\delta_\alpha^\beta, \quad |p^I][p_I| = m\delta_{\dot{\alpha}}^{\dot{\beta}},\end{aligned} \tag{367}$$

$$p|p^I] = -m|p^I\rangle, \quad p|p_I\rangle = -m|p_I]. \tag{368}$$

For on-shell three-point amplitudes, the following conditions must be satisfied:

$$\begin{aligned}&p_3 = p_1+p_2, \quad (p_i)_{\alpha\dot{\alpha}}(p_j)^{\dot{\alpha}\beta}+(p_j)_{\alpha\dot{\alpha}}(p_i)^{\dot{\alpha}\beta} = 2(p_i\cdot p_j)\delta_\alpha^\beta,\\ &p_i^2 = m_i^2 \quad (i,j=1,2,3).\end{aligned} \tag{369}$$

Here, we choose an arbitrary reference frame and define the canonical-spinor variables for particle-1, particle-2, and particle-3 as follows:

$$|1_I\rangle = \sqrt{m_1}\begin{pmatrix} c_1^2 e^{\frac{\eta_1}{2}} + |s_1|^2 e^{-\frac{\eta_1}{2}} & c_1 s_1^*(e^{\frac{\eta_1}{2}} - e^{-\frac{\eta_1}{2}}) \\ c_1 s_1(e^{\frac{\eta_1}{2}} - e^{-\frac{\eta_1}{2}}) & |s_1|^2 e^{\frac{\eta_1}{2}} + c_1^2 e^{-\frac{\eta_1}{2}} \end{pmatrix}_{\alpha I} = \begin{pmatrix} A_1 & B_1 \\ B_1^* & C_1 \end{pmatrix}_{\alpha I},$$

$$[1^I| = \sqrt{m_1}\begin{pmatrix} c_1^2 e^{\frac{\eta_1}{2}} + |s_1|^2 e^{-\frac{\eta_1}{2}} & c_1 s_1^*(e^{\frac{\eta_1}{2}} - e^{-\frac{\eta_1}{2}}) \\ c_1 s_1(e^{\frac{\eta_1}{2}} - e^{-\frac{\eta_1}{2}}) & |s_1|^2 e^{\frac{\eta_1}{2}} + c_1^2 e^{-\frac{\eta_1}{2}} \end{pmatrix}^I_{\dot\alpha} = \begin{pmatrix} A_1 & B_1 \\ B_1^* & C_1 \end{pmatrix}^I_{\dot\alpha},$$

$$|2_I\rangle = \sqrt{m_2}\begin{pmatrix} c_2^2 e^{\frac{\eta_2}{2}} + |s_2|^2 e^{-\frac{\eta_2}{2}} & c_2 s_2^*(e^{\frac{\eta_2}{2}} - e^{-\frac{\eta_2}{2}}) \\ c_2 s_2(e^{\frac{\eta_2}{2}} - e^{-\frac{\eta_2}{2}}) & |s_2|^2 e^{\frac{\eta_2}{2}} + c_2^2 e^{-\frac{\eta_2}{2}} \end{pmatrix}_{\alpha I} = \begin{pmatrix} A_2 & B_2 \\ B_2^* & C_2 \end{pmatrix}_{\alpha I},$$

$$[2^I| = \sqrt{m_2}\begin{pmatrix} c_2^2 e^{\frac{\eta_2}{2}} + |s_2|^2 e^{-\frac{\eta_2}{2}} & c_2 s_2^*(e^{\frac{\eta_2}{2}} - e^{-\frac{\eta_2}{2}}) \\ c_2 s_2(e^{\frac{\eta_2}{2}} - e^{-\frac{\eta_2}{2}}) & |s_2|^2 e^{\frac{\eta_2}{2}} + c_2^2 e^{-\frac{\eta_2}{2}} \end{pmatrix}^I_{\dot\alpha} = \begin{pmatrix} A_2 & B_2 \\ B_2^* & C_2 \end{pmatrix}^I_{\dot\alpha},$$

$$|3_I\rangle = \sqrt{m_3}\begin{pmatrix} c_3^2 e^{\frac{\eta_3}{2}} + |s_3|^2 e^{-\frac{\eta_3}{2}} & c_3 s_3^*(e^{\frac{\eta_3}{2}} - e^{-\frac{\eta_3}{2}}) \\ c_3 s_3(e^{\frac{\eta_3}{2}} - e^{-\frac{\eta_3}{2}}) & |s_3|^2 e^{\frac{\eta_3}{2}} + c_3^2 e^{-\frac{\eta_3}{2}} \end{pmatrix}_{\alpha I} = \begin{pmatrix} A_3 & B_3 \\ B_3^* & C_3 \end{pmatrix}_{\alpha I},$$

$$[3^I| = \sqrt{m_3}\begin{pmatrix} c_3^2 e^{\frac{\eta_3}{2}} + |s_3|^2 e^{-\frac{\eta_3}{2}} & c_3 s_3^*(e^{\frac{\eta_3}{2}} - e^{-\frac{\eta_3}{2}}) \\ c_3 s_3(e^{\frac{\eta_3}{2}} - e^{-\frac{\eta_3}{2}}) & |s_3|^2 e^{\frac{\eta_3}{2}} + c_3^2 e^{-\frac{\eta_3}{2}} \end{pmatrix}^I_{\dot\alpha} = \begin{pmatrix} A_3 & B_3 \\ B_3^* & C_3 \end{pmatrix}^I_{\dot\alpha}, \tag{370}$$

with

$$A_j = c_j^2 e^{\frac{\eta_j}{2}} + |s_j|^2 e^{-\frac{\eta_j}{2}}, \quad B_j = c_j s_j^*(e^{\frac{\eta_j}{2}} - e^{-\frac{\eta_j}{2}}), \\ C_j = |s_j|^2 e^{\frac{\eta_j}{2}} + c_j^2 e^{-\frac{\eta_j}{2}}, \tag{371}$$

where $c_j = \cos\dfrac{\theta_j}{2}, s_j = e^{i\phi_j}\sin\dfrac{\theta_j}{2}, \eta_j = \operatorname{arctanh}\dfrac{|\mathbf{p}_j|}{E_j}$ $(j=1,2,3)$. Using the antisymmetric tensors $\varepsilon_{\alpha\beta}$ and ε_{IJ} (and their inverses) to raise and lower Lorentz and little group indices, the remaining spinors follow straightforwardly from the above definitions.

As an illustration, the amplitude in Eq. (376) can be written in explicit form as

$$\mathcal{A}^{\{I_1^{(1)}\}}_{\{I_1^{(3)}\}}(0,\frac{1}{2}) \propto \langle 1^{I_1^{(1)}} 3_{I_1^{(3)}}\rangle - [1^{I_1^{(1)}} 3_{I_1^{(3)}}] = \begin{pmatrix} C_1 & -B_1 \\ -B_1^* & A_1 \end{pmatrix}\begin{pmatrix} A_3 & B_3 \\ B_3^* & C_3 \end{pmatrix} - \begin{pmatrix} A_1 & B_1 \\ B_1^* & C_1 \end{pmatrix}\begin{pmatrix} -C_3 & B_3 \\ B_3^* & -A_3 \end{pmatrix} \\ = \begin{pmatrix} C_1A_3 + A_1C_3 - 2B_1B_3^* & C_1B_3 - B_1C_3 - A_1B_3 + B_1A_3 \\ A_1B_3^* - B_1^*A_3 + B_1^*C_3 - C_1B_3^* & A_1C_3 + C_1A_3 - 2B_1^*B_3 \end{pmatrix}^{I_1^{(1)}}_{I_1^{(3)}}. \tag{372}$$

In what follows, canonical-spinor variables are denoted in **BOLD** for brevity, and the $SU(2)$ little group indices are suppressed, as they are totally symmetric and would make the notation lengthy.

1. $s_3 = \frac{1}{2}, s_1 = \frac{1}{2}, s_2 = 0$. There are two (L,S) combinations: $(0,\frac{1}{2})$ and $(1,\frac{1}{2})$.

(a) $L=0, S=\frac{1}{2}$:

$$\mathcal{A}^{\{I_1^{(1)}\}}_{\{I_1^{(3)}\}}(0,\frac{1}{2}) = \mathcal{S}^{\{I_1^{(1)}\}}_{\{K_1\}}\mathcal{Y}^0 C^{\frac{1}{2},\{K_1\};0}_{\frac{1}{2},\{I_1^{(3)}\}}, \tag{373}$$

where

$$\mathcal{S}^{\{I_1^{(1)}\}}_{\{K_1\}} = \frac{1}{\sqrt{m_3(E_{1\text{com}}+m_1)}}(\tau_1^1)^{\{I_1^{(1)}\}}_{\{J_1\}} C^{\frac{1}{2},\{J_1\};0}_{\frac{1}{2},\{K_1\}} \\ = \frac{1}{\sqrt{m_3(E_{1\text{com}}+m_1)}}(\langle\mathbf{13}\rangle - [\mathbf{13}]), \tag{374}$$

and

$$\mathcal{Y}^0 = \sqrt{\frac{m_3}{\pi|\mathbf{p}_{1\text{com}}|}}. \tag{375}$$

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}\}}_{\{I_1^{(3)}\}}(0,\frac{1}{2}) = \sqrt{\frac{m_3}{\pi|\mathbf{p}_{1\text{com}}|}}\frac{1}{\sqrt{m_3(E_{1\text{com}}+m_1)}}(\langle\mathbf{13}\rangle - [\mathbf{13}]). \tag{376}$$

(b) $L=1, S=\frac{1}{2}$:

$$\mathcal{A}^{\{I_1^{(1)}\}}_{\{I_1^{(3)}\}}(1,\frac{1}{2})=\mathcal{S}^{\{I_1^{(1)}\}}_{\{K_1\}}\mathcal{Y}^1_{\{L_1L_2\}}C^{\frac{1}{2},\{K_1\};1,\{L_1L_2\}}_{\frac{1}{2},\{I_1^{(3)}\}}, \tag{377}$$

where $\mathcal{S}^{\{I_1^{(1)}\}}_{\{K_1\}}$ can be obtained from Eq. (374), and

$$\mathcal{Y}^1_{\{L_1L_2\}}=\sqrt{\frac{3}{2\pi|\mathbf{p}_{1\text{com}}|^3m_3}}\langle\mathbf{3}|p_1|\mathbf{3}]. \tag{378}$$

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}\}}_{\{I_1^{(3)}\}}(1,\frac{1}{2})=\sqrt{\frac{3}{2\pi|\mathbf{p}_{1\text{com}}|^3m_3}}\frac{(m_1+m_2+m_3)(m_3+m_1-m_2)}{2\sqrt{m_3(E_{1\text{com}}+m_1)}}\times\sqrt{\frac{2}{3}}\times(\langle\mathbf{13}\rangle+[\mathbf{13}]). \tag{379}$$

2. $s_3=0, s_1=\frac{1}{2}, s_2=\frac{1}{2}$. There are two (L,S) combinations: $(0,0)$ and $(1,1)$.

(a) $L=0, S=0$:

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}(0,0)=\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}\mathcal{Y}^0C^{0;0}_0, \tag{380}$$

where $\mathcal{Y}^0$ can be obtained from Eq. (375), and

$$\begin{aligned}\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}&=\frac{1}{\sqrt{m_3(E_{1\text{com}}+m_1)m_3(E_{2\text{com}}+m_2)}}\times(\tau_1^1)^{\{I_1^{(1)}\}}_{\{J_1\}}(\tau_2^1)^{\{I_1^{(2)}\}}_{\{N_1\}}C^{\frac{1}{2},\{J_1\};\frac{1}{2},\{N_1\}}_0\\&=\sqrt{\frac{1}{2}}\frac{m_1+m_2+m_3}{\sqrt{m_3(E_{1\text{com}}+m_1)m_3(E_{2\text{com}}+m_2)}}\times(\langle\mathbf{12}\rangle-[\mathbf{12}]).\end{aligned} \tag{381}$$

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}(0,0)=\sqrt{\frac{m_3}{\pi|\mathbf{p}_{1\text{com}}|}}\sqrt{\frac{1}{2}}\times\frac{m_1+m_2+m_3}{\sqrt{m_3(E_{1\text{com}}+m_1)m_3(E_{2\text{com}}+m_2)}}\times(\langle\mathbf{12}\rangle-[\mathbf{12}]). \tag{382}$$

(b) $L=1, S=1$:

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}(1,1)=\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{K_1K_2\}}\mathcal{Y}^1_{\{L_1L_2\}}C^{1,\{K_1K_2\};1,\{L_1L_2\}}_0, \tag{383}$$

where $\mathcal{Y}^1_{\{L_1L_2\}}$ can be obtained from Eq. (378) and

$$\begin{aligned}\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{K_1K_2\}}&=\frac{1}{\sqrt{m_3(E_{1\text{com}}+m_1)m_3(E_{2\text{com}}+m_2)}}\times(\tau_1^1)^{\{I_1^{(1)}\}}_{\{J_1\}}(\tau_2^1)^{\{I_1^{(2)}\}}_{\{N_1\}}C^{\frac{1}{2},\{J_1\};\frac{1}{2},\{N_1\}}_{1,\{K_1K_2\}}\\&=\frac{1}{\sqrt{m_3(E_{1\text{com}}+m_1)m_3(E_{2\text{com}}+m_2)}}\times(\langle\mathbf{13}\rangle-[\mathbf{13}])(\langle\mathbf{23}\rangle-[\mathbf{23}]).\end{aligned} \tag{384}$$

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}(1,1)=\sqrt{\frac{3}{2\pi|\mathbf{p}_{1\text{com}}|^3m_3}}\times\frac{-(m_1+m_2+m_3)(m_3^2-(m_1-m_2)^2)}{\sqrt{m_3(E_{1\text{com}}+m_1)m_3(E_{2\text{com}}+m_2)}}\sqrt{\frac{1}{3}}\times(\langle\mathbf{12}\rangle+[\mathbf{12}]). \tag{385}$$

3. $s_3=1, s_1=0, s_2=0$. There is one (L,S) combination: $(1,0)$.

(a) $L=1, S=0$:

$$\mathcal{A}_{\{I_1^{(3)}I_2^{(3)}\}}(1,0)=\mathcal{Y}^1_{\{L_1L_2\}}C^{0;1,\{L_1L_2\}}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{386}$$

where $\mathcal{Y}^1_{\{L_1L_2\}}$ can be obtained from Eq. (378).

The amplitude is

$$\mathcal{A}_{\{I_1^{(3)}I_2^{(3)}\}}(1,0)=\sqrt{\frac{3}{2\pi|\mathbf{p}_{1\text{com}}|^3m_3}}\langle\mathbf{3}|p_1|\mathbf{3}]. \tag{387}$$

4. $s_3=0, s_1=1, s_2=0$. There is one (L,S) combination: $(1,1)$.

(a) $L=1, S=1$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\}}(1,1)=\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{K_1K_2\}}\mathcal{Y}^1_{\{L_1L_2\}}C^{1,\{K_1K_2\};1,\{L_1L_2\}}_0, \tag{388}$$

where $\mathcal{Y}^1_{\{L_1L_2\}}$ can be obtained from Eq. (378), and

$$\begin{aligned}\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{K_1K_2\}}&=\frac{1}{m_3(E_{1\text{com}}+m_1)}(\tau_1^2)^{\{I_1^{(1)}I_2^{(1)}\}}_{\{J_1J_2\}}C^{1,\{J_1J_2\};0}_{1,\{K_1K_2\}}\\&=\frac{1}{m_3(E_{1\text{com}}+m_1)}(\langle\mathbf{13}\rangle-[\mathbf{13}])^2.\end{aligned} \tag{389}$$

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\}}(1,1)=\sqrt{\frac{3}{2\pi|\mathbf{p}_{1\text{com}}|^3m_3}}\times\frac{-(m_1+m_2+m_3)(m_3+m_1-m_2)}{m_3(E_{1\text{com}}+m_1)}\times\sqrt{\frac{1}{3}}\langle\mathbf{1}|p_3|\mathbf{1}]. \tag{390}$$

5. $s_3=1, s_1=1, s_2=0$. There are three (L,S) combinations: $(0,1)$, $(1,1)$, and $(2,1)$.

(a) $L=0, S=1$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(0,1)=\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{K_1K_2\}}\mathcal{Y}^0C^{1,\{K_1K_2\};0}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{391}$$

where $\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{K_1K_2\}}$ and $\mathcal{Y}^0$ can be obtained from Eqs. (389) and (375), respectively.

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(0,1)=\sqrt{\frac{m_3}{\pi|\mathbf{p}_{1\rm com}|}}\frac{1}{m_3(E_{1\rm com}+m_1)}(\langle\mathbf{13}\rangle-[\mathbf{13}])^2. \tag{392}$$

(b) $L=1, S=1$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(1,1)=\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{K_1K_2\}}\mathcal{Y}^1_{\{L_1L_2\}}C^{1,\{K_1K_2\};1,\{L_1L_2\}}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{393}$$

where $\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{K_1K_2\}}$ and $\mathcal{Y}^1_{\{L_1L_2\}}$ can be obtained from Eqs. (389) and (378), respectively.

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(1,1)=\sqrt{\frac{3}{2\pi|\mathbf{p}_{1\rm com}|^3m_3}}\frac{(m_1+m_2+m_3)(m_3+m_1-m_2)}{2m_3(E_{1\rm com}+m_1)}\times(\langle\mathbf{13}\rangle^2-[\mathbf{13}]^2). \tag{394}$$

(b) $L=2, S=1$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(2,1)=\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{K_1K_2\}}\mathcal{Y}^2_{\{L_1L_2L_3L_4\}}C^{1,\{K_1K_2\};2,\{L_1L_2L_3L_4\}}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{395}$$

where $\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{K_1K_2\}}$ can be obtained from Eq. (389), and

$$\mathcal{Y}^2_{\{L_1L_2L_3L_4\}}=\sqrt{\frac{15}{8\pi|\mathbf{p}_{1\rm com}|^5m_3^3}}(\langle\mathbf{3}|p_1|\mathbf{3}])^2. \tag{396}$$

The amplitude is

$$\begin{aligned}\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(2,1)=&\sqrt{\frac{15}{8\pi|\mathbf{p}_{1\rm com}|^5m_3^3}}\frac{(m_1+m_2+m_3)(m_3+m_1-m_2)}{\sqrt{m_3(E_{1\rm com}+m_1)m_3(E_{2\rm com}+m_2)}}\sqrt{\frac{3}{5}}\\&\times\left(\frac{(m_1+m_2+m_3)(m_3+m_1-m_2)}{4}(\langle\mathbf{13}\rangle+[\mathbf{13}])^2-\langle\mathbf{1}|p_3|\mathbf{1}]\langle\mathbf{3}|p_1|\mathbf{3}]\right).\end{aligned} \tag{397}$$

6. $s_3=0, s_1=1, s_2=1$. There are three (L,S) combinations: $(0,0)$, $(1,1)$, and $(2,2)$.

(a) $L=0, S=0$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}(0,0)=\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}\mathcal{Y}^0C^{0;0}_0, \tag{398}$$

where $\mathcal{Y}^0$ can be obtained from Eq. (375).

$$\begin{aligned}\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}&=\frac{1}{(m_3(E_{1\rm com}+m_1))(m_3(E_{2\rm com}+m_2))}(\tau_1^2)^{\{I_1^{(1)}I_2^{(1)}\}}_{\{J_1J_2\}}(\tau_2^2)^{\{I_1^{(2)}I_2^{(2)}\}}_{\{N_1N_2\}}\times C_0^{1,\{J_1J_2\};1,\{N_1N_2\}}\\&=\sqrt{\frac{1}{3}}\frac{(m_1+m_2+m_3)^2}{(m_3(E_{1\rm com}+m_1))(m_3(E_{2\rm com}+m_2))}(\langle\mathbf{12}\rangle-[\mathbf{12}])^2.\end{aligned} \tag{399}$$

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}(0,0)=\sqrt{\frac{m_3}{\pi|\mathbf{p}_{1\rm com}|}}\sqrt{\frac{1}{3}}\frac{(m_1+m_2+m_3)^2}{(m_3(E_{1\rm com}+m_1))(m_3(E_{2\rm com}+m_2))}(\langle\mathbf{12}\rangle-[\mathbf{12}])^2. \tag{400}$$

(b) $L=1, S=1$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}(1,1)=\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2\}}\mathcal{Y}^1_{\{L_1L_2\}}C_0^{1,\{K_1K_2\};1,\{L_1L_2\}}, \tag{401}$$

where $\mathcal{Y}^1_{\{L_1L_2\}}$ can be obtained from Eq. (378) and

$$\begin{aligned}\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2\}} &= \frac{1}{(m_3(E_{1\mathrm{com}}+m_1))(m_3(E_{2\mathrm{com}}+m_2))}(\tau_1^2)^{\{I_1^{(1)}I_2^{(1)}\}}_{\{J_1J_2\}}(\tau_2^2)^{\{I_1^{(2)}I_2^{(2)}\}}_{\{N_1N_2\}}C^{1,\{J_1J_2\};1,\{N_1N_2\}}_{1,\{K_1K_2\}}\\ &= \frac{(m_1+m_2+m_3)}{(m_3(E_{1\mathrm{com}}+m_1))(m_3(E_{2\mathrm{com}}+m_2))}\times(\langle\mathbf{12}\rangle-[\mathbf{12}])(\langle\mathbf{13}\rangle-[\mathbf{13}])(\langle\mathbf{23}\rangle-[\mathbf{23}]).\end{aligned}\tag{402}$$

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}(1,1)=\sqrt{\frac{3}{2\pi|\mathbf{p}_{1\mathrm{com}}|^3m_3}}\frac{-(m_1+m_2+m_3)^2(m_3^2-(m_1-m_2)^2)}{(m_3(E_{1\mathrm{com}}+m_1))(m_3(E_{2\mathrm{com}}+m_2))}\sqrt{\frac{1}{3}}\times(\langle\mathbf{12}\rangle^2-[\mathbf{12}]^2).\tag{403}$$

(c) $L=2, S=2$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}(2,2)=\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2K_3K_4\}}\mathcal{Y}^2_{\{L_1L_2L_3L_4\}}C_0^{2,\{K_1K_2K_3K_4\};2,\{L_1L_2L_3L_4\}},\tag{404}$$

where $\mathcal{Y}^2_{\{L_1L_2L_3L_4\}}$ can be obtained from Eq. (396), and

$$\begin{aligned}\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2K_3K_4\}} &= \frac{1}{(m_3(E_{1\mathrm{com}}+m_1))(m_3(E_{2\mathrm{com}}+m_2))}(\tau_1^2)^{\{I_1^{(1)}I_2^{(1)}\}}_{\{J_1J_2\}}(\tau_2^2)^{\{I_1^{(2)}I_2^{(2)}\}}_{\{N_1N_2\}}\times C^{1,\{J_1J_2\};1,\{N_1N_2\}}_{2,\{K_1K_2K_3K_4\}}\\ &= \frac{1}{(m_3(E_{1\mathrm{com}}+m_1))(m_3(E_{2\mathrm{com}}+m_2))}\times(\langle\mathbf{13}\rangle-[\mathbf{13}])^2(\langle\mathbf{23}\rangle-[\mathbf{23}])^2.\end{aligned}\tag{405}$$

The amplitude is

$$\begin{aligned}\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}(2,2) &= \sqrt{\frac{15}{8\pi|\mathbf{p}_{1\mathrm{com}}|^5m_3^3}}\frac{(m_1+m_2+m_3)^2(m_3^2-(m_1-m_2)^2)}{(m_3(E_{1\mathrm{com}}+m_1))(m_3(E_{2\mathrm{com}}+m_2))}\sqrt{\frac{1}{5}}\\ &\times\Big(((m_1-m_2)^2-m_3^2)(\langle\mathbf{12}\rangle+[\mathbf{12}])^2+\langle\mathbf{1}|p_3|\mathbf{1}]\langle\mathbf{2}|p_3|\mathbf{2}]\Big).\end{aligned}\tag{406}$$

7. $s_3=\frac{1}{2}$, $s_1=\frac{3}{2}$, $s_2=0$. There are two (L,S) combinations: $(1,\frac{3}{2})$ and $(2,\frac{3}{2})$.

(a) $L=1, S=\frac{3}{2}$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}I_3^{(1)}\}}_{\{I_1^{(3)}\}}(1,\frac{3}{2})=\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}I_3^{(1)}\}}_{\{K_1K_2K_3\}}\mathcal{Y}^1_{\{L_1L_2\}}C^{\frac{3}{2},\{K_1K_2K_3\};1,\{L_1L_2\}}_{\frac{1}{2},\{I_1^{(3)}\}}.\tag{407}$$

where $\mathcal{Y}^1_{\{L_1L_2\}}$ can be obtained from Eq. (378), and

$$\begin{aligned}\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}I_3^{(1)}\}}_{\{K_1K_2K_3\}} &= \frac{1}{(m_3(E_{1\mathrm{com}}+m_1))^{\frac{3}{2}}}(\tau_1^3)^{\{I_1^{(1)}I_2^{(1)}I_3^{(1)}\}}_{\{J_1J_2J_3\}}C^{\frac{3}{2},\{J_1J_2J_3\};0}_{\frac{3}{2},\{K_1K_2K_3\}}\\ &= \frac{1}{(m_3(E_{1\mathrm{com}}+m_1))^{\frac{3}{2}}}(\langle\mathbf{13}\rangle-[\mathbf{13}])^3.\end{aligned}\tag{408}$$

The amplitude is

$$\begin{aligned}&\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}I_3^{(1)}\}}_{\{I_1^{(3)}\}}(1,\frac{3}{2})\\ &= \sqrt{\frac{3}{2\pi|\mathbf{p}_{1\mathrm{com}}|^3m_3}}\frac{-(m_1+m_2+m_3)(m_3+m_1-m_2)}{(m_3(E_{1\mathrm{com}}+m_1))^{\frac{3}{2}}}.\\ &\times\sqrt{\frac{1}{2}}\times(\langle\mathbf{13}\rangle-[\mathbf{13}])\langle\mathbf{1}|p_3|\mathbf{1}].\end{aligned}\tag{409}$$

(b) $L=2, S=\frac{3}{2}$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}I_3^{(1)}\}}_{\{I_1^{(3)}\}}(2,\frac{3}{2})=\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}I_3^{(1)}\}}_{\{K_1K_2K_3\}}\mathcal{Y}^2_{\{L_1L_2L_3L_4\}}C^{\frac{3}{2},\{K_1K_2K_3\};2,\{L_1L_2L_3L_4\}}_{\frac{1}{2},\{I_1^{(3)}\}}.\tag{410}$$

where $\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}I_3^{(1)}\}}_{\{K_1K_2K_3\}}$ and $\mathcal{Y}^2_{\{L_1L_2L_3L_4\}}$ can be obtained from Eqs. (408) and (396), respectively.

The amplitude is

$$\begin{aligned}&\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}I_3^{(1)}\}}_{\{I_1^{(3)}\}}(2,\frac{3}{2})\\ &= \sqrt{\frac{15}{8\pi|\mathbf{p}_{1\mathrm{com}}|^5m_3^3}}\frac{-(m_1+m_2+m_3)^2(m_3+m_1-m_2)^2}{2(m_3(E_{1\mathrm{com}}+m_1))^{\frac{3}{2}}}\sqrt{\frac{2}{5}}\\ &\times(\langle\mathbf{13}\rangle+[\mathbf{13}])\langle\mathbf{1}|p_3|\mathbf{1}].\end{aligned}\tag{411}$$

8. $s_3=\frac{1}{2}$, $s_1=\frac{1}{2}$, $s_2=1$. There are four (L,S) com-

binations: $(0,\frac{1}{2})$, $(1,\frac{1}{2})$, $(1,\frac{3}{2})$, and $(2,\frac{3}{2})$.

(a) $L=0, S=\frac{1}{2}$

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}\}}(0,\frac{1}{2})=\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1\}}\mathcal{Y}^0 C^{\frac{1}{2},\{K_1\};0}_{\frac{1}{2},\{I_1^{(3)}\}}, \tag{412}$$

where $\mathcal{Y}^0$ can be obtained from Eq. (375), and

$$\begin{aligned}\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1\}}&=\frac{1}{\sqrt{m_3(E_{1\rm com}+m_1)(m_3(E_{2\rm com}+m_2))^2}}(\tau_1^1)^{\{I_1^{(1)}\}}_{\{J_1\}}\\&\quad\times(\tau_2^2)^{\{I_1^{(2)}I_2^{(2)}\}}_{\{N_1N_2\}}\times C^{\frac{1}{2},\{J_1\};1,\{N_1N_2\}}_{\frac{1}{2},\{K_1\}}\\&=\sqrt{\frac{2}{3}}\frac{(m_1+m_2+m_3)}{\sqrt{m_3(E_{1\rm com}+m_1)(m_3(E_{2\rm com}+m_2))^2}}\\&\quad\times(\langle\mathbf{23}\rangle-[\mathbf{23}])(\langle\mathbf{12}\rangle-[\mathbf{12}]).\end{aligned} \tag{413}$$

The amplitude is

$$\begin{aligned}&\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}\}}(0,\frac{1}{2})\\&=\sqrt{\frac{m_3}{\pi|\mathbf{p}_{1\rm com}|}}\sqrt{\frac{2}{3}}\frac{(m_1+m_2+m_3)}{\sqrt{m_3(E_{1\rm com}+m_1)(m_3(E_{2\rm com}+m_2))^2}}\\&\times(\langle\mathbf{23}\rangle-[\mathbf{23}])(\langle\mathbf{12}\rangle-[\mathbf{12}]).\end{aligned} \tag{414}$$

(b) $L=1, S=\frac{1}{2}$:

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}\}}(1,\frac{1}{2})=\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1\}}\mathcal{Y}^1_{\{L_1L_2\}} C^{\frac{1}{2},\{K_1\};1,\{L_1L_2\}}_{\frac{1}{2},\{I_1^{(3)}\}}, \tag{415}$$

where $\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)},I_2^{(2)}\}}_{\{K_1\}}$ and $\mathcal{Y}^1_{\{L_1,L_2\}}$ can be obtained from Eq. (413) and Eq. (378), respectively.

The amplitude is

$$\begin{aligned}&\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}\}}(1,\frac{1}{2})\\&=\sqrt{\frac{3}{2\pi|\mathbf{p}_{1\rm com}|^3m_3}}\frac{-(m_1+m_2+m_3)^2(m_3+m_2-m_1)}{\sqrt{m_3(E_{1\rm com}+m_1)(m_3(E_{2\rm com}+m_2))^2}}\frac{2}{3}\\&\times(\langle\mathbf{23}\rangle+[\mathbf{23}])(\langle\mathbf{12}\rangle-[\mathbf{12}]).\end{aligned} \tag{416}$$

(c) $L=1, S=\frac{3}{2}$:

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}\}}(1,\frac{3}{2})=\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2K_3\}}\mathcal{Y}^1_{\{L_1L_2\}} C^{\frac{3}{2},\{K_1K_2K_3\};1,\{L_1L_2\}}_{\frac{1}{2},\{I_1^{(3)}\}}, \tag{417}$$

where $\mathcal{Y}^1_{\{L_1L_2\}}$ can be obtained from Eq. (378), and

$$\begin{aligned}\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2K_3\}}&=\frac{1}{\sqrt{m_3(E_{1\rm com}+m_1)(m_3(E_{2\rm com}+m_2))^2}}(\tau_1^1)^{\{I_1^{(1)}\}}_{\{J_1\}}\\&\quad\times(\tau_2^2)^{\{I_1^{(2)}I_2^{(2)}\}}_{\{N_1N_2\}}\times C^{\frac{1}{2},\{J_1\};1,\{N_1N_2\}}_{\frac{3}{2},\{K_1K_2K_3\}}\\&=\frac{1}{\sqrt{m_3(E_{1\rm com}+m_1)(m_3(E_{2\rm com}+m_2))^2}}\\&\quad\times(\langle\mathbf{13}\rangle-[\mathbf{13}])(\langle\mathbf{23}\rangle-[\mathbf{23}])^2.\end{aligned} \tag{418}$$

The amplitude is

$$\begin{aligned}&\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}\}}(1,\frac{3}{2})\\&=\sqrt{\frac{3}{2\pi|\mathbf{p}_{1\rm com}|^3m_3}}\frac{(m_1+m_2+m_3)(m_3+m_2-m_1)}{\sqrt{m_3(E_{1\rm com}+m_1)}(m_3(E_{2\rm com}+m_2))}\sqrt{\frac{1}{2}}\\&\times\Big((m_2-m_3-m_1)([\mathbf{12}]+\langle\mathbf{12}\rangle)(\langle\mathbf{23}\rangle-[\mathbf{23}])\\&\quad+\langle\mathbf{2}|p_1|\mathbf{2}](\langle\mathbf{13}\rangle-[\mathbf{13}])\Big).\end{aligned} \tag{419}$$

(d) $L=2, S=\frac{3}{2}$:

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}\}}(2,\frac{3}{2})=\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2K_3\}}\mathcal{Y}^2_{\{L_1L_2L_3L_4\}} C^{\frac{3}{2},\{K_1K_2K_3\};2,\{L_1L_2L_3L_4\}}_{\frac{1}{2},\{I_1^{(3)}\}}, \tag{420}$$

where $\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2K_3\}}$ and $\mathcal{Y}^2_{\{L_1L_2L_3L_4\}}$ can be obtained from Eq. (418) and Eq. (396), respectively. The amplitude is

$$\begin{aligned}&\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}\}}(2,\frac{3}{2})\\&=\sqrt{\frac{15}{8\pi|\mathbf{p}_{1\rm com}|^5m_3^3}}\frac{(m_1+m_2+m_3)^2(m_3^2-(m_2-m_1)^2)}{\sqrt{(m_3(E_{1\rm com}+m_1))}m_3(E_{2\rm com}+m_2)}\sqrt{\frac{2}{5}}\\&\times\Big((m_3+m_2-m_1)([\mathbf{12}]+\langle\mathbf{12}\rangle)([\mathbf{23}]+\langle\mathbf{23}\rangle)\\&\quad+\langle\mathbf{2}|1|\mathbf{2}]([\mathbf{13}]+\langle\mathbf{13}\rangle)\Big).\end{aligned} \tag{421}$$

9. $s_3=1$, $s_1=\frac{1}{2}$, $s_2=\frac{1}{2}$. There are four (L,S) combinations: $(1,0)$, $(0,1)$, $(1,1)$, and $(2,1)$.

(a) $L=1, S=0$:

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(1,0)=\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}\mathcal{Y}^1_{\{L_1L_2\}} C^{0;1,\{L_1L_2\}}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{422}$$

where $\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}$ and $\mathcal{Y}^1_{\{L_1L_2\}}$ can be obtained from Eqs. (381) and (378), respectively. The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(1,0) = \sqrt{\frac{3}{2\pi|\mathbf{p}_{1\rm com}|^3 m_3}}\sqrt{\frac{1}{2}}\frac{m_1+m_2+m_3}{\sqrt{m_3(E_{1\rm com}+m_1)m_3(E_{2\rm com}+m_2)}} \times(\langle\mathbf{12}\rangle-[\mathbf{12}])\langle\mathbf{3}|p_1|\mathbf{3}]. \tag{423}$$

(b) $L=0, S=1$:

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(0,1) = \mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{K_1K_2\}}\mathcal{Y}^0 C^{1,\{K_1K_2\};0}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{424}$$

where $\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{K_1K_2\}}$ and $\mathcal{Y}^0$ can be obtained from Eqs. (384) and (375), respectively.

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(0,1) = \sqrt{\frac{m_3}{\pi|\mathbf{p}_{1\rm com}|}}\frac{1}{\sqrt{m_3(E_{1\rm com}+m_1)m_3(E_{2\rm com}+m_2)}} \times(\langle\mathbf{13}\rangle-[\mathbf{13}])(\langle\mathbf{23}\rangle-[\mathbf{23}]). \tag{425}$$

(c) $L=1, S=1$:

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(1,1) = \mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{K_1K_2\}}\mathcal{Y}^1_{\{L_1L_2\}} C^{1,\{K_1K_2\};1,\{L_1L_2\}}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{426}$$

where $\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{K_1K_2\}}$ and $\mathcal{Y}^1_{\{L_1L_2\}}$ can be obtained from Eqs. (384) and (378), respectively.

The amplitude is

$$\begin{aligned}\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(1,1) &= \sqrt{\frac{3}{2\pi|\mathbf{p}_{1\rm com}|^3 m_3}}\frac{(m_1+m_2+m_3)}{2\sqrt{m_3(E_{1\rm com}+m_1)m_3(E_{2\rm com}+m_2)}} \\ &\times\Big((m_3+m_1-m_2)(\langle\mathbf{13}\rangle+[\mathbf{13}])(\langle\mathbf{23}\rangle-[\mathbf{23}]) \\ &-(m_3+m_2-m_1)(\langle\mathbf{13}\rangle-[\mathbf{13}])(\langle\mathbf{23}\rangle+[\mathbf{23}])\Big).\end{aligned} \tag{427}$$

(d) $L=2, S=1$:

$$\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(2,1) = \mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{K_1K_2\}}\mathcal{Y}^2_{\{L_1L_2L_3L_4\}} C^{1,\{K_1K_2\};2,\{L_1L_2L_3L_4\}}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{428}$$

where $\mathcal{S}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{K_1K_2\}}$ and $\mathcal{Y}^2_{\{L_1L_2L_3L_4\}}$ can be obtained from Eqs. (384) and (396), respectively.

The amplitude is

$$\begin{aligned}\mathcal{A}^{\{I_1^{(1)}\},\{I_1^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(2,1) &= \sqrt{\frac{m_3}{\pi|\mathbf{p}_{1\rm com}|}}\frac{-(m_1+m_2+m_3)(m_3^2-(m_1^2-m_2^2))}{\sqrt{m_3(E_{1\rm com}+m_1)m_3(E_{2\rm com}+m_2)}} \\ &\times\Big((\langle\mathbf{12}\rangle+[\mathbf{12}])\langle\mathbf{3}|p_1|\mathbf{3}] \\ &+\frac{(m_1+m_2+m_3)}{4}(\langle\mathbf{13}\rangle+[\mathbf{13}])(\langle\mathbf{23}\rangle+[\mathbf{23}]).\end{aligned} \tag{429}$$

10. $s_3=1, s_1=1, s_2=1$. There are seven (L,S) combinations: $(1,0)$, $(0,1)$, $(1,1)$, $(2,1)$, $(1,2)$, $(2,2)$, and $(3,2)$.

(a) $L=1, S=0$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(1,0) = \mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}\mathcal{Y}^1_{\{L_1L_2\}} C^{0;1,\{L_1L_2\}}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{430}$$

where $\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}$ and $\mathcal{Y}^1_{\{L_1L_2\}}$ can be obtained from Eqs. (399) and (378), respectively.

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(1,0) = \sqrt{\frac{3}{2\pi|\mathbf{p}_{1\rm com}|^3 m_3}}\sqrt{\frac{1}{3}}\frac{(m_1+m_2+m_3)^2}{(m_3(E_{1\rm com}+m_1))(m_3(E_{2\rm com}+m_2))} \times(\langle\mathbf{12}\rangle-[\mathbf{12}])^2\langle\mathbf{3}|p_1|\mathbf{3}]. \tag{431}$$

(b) $L=0, S=1$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(0,1) = \mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2\}}\mathcal{Y}^0 C^{1\{K_1K_2\};0}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{432}$$

where $\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2\}}$ and $\mathcal{Y}^0$ can be obtained from Eqs. (402) and (375), respectively.

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(0,1) = \sqrt{\frac{m_3}{\pi|\mathbf{p}_{1\rm com}|}}\frac{(m_1+m_2+m_3)}{(m_3(E_{1\rm com}+m_1))(m_3(E_{2\rm com}+m_2))} \times(\langle\mathbf{12}\rangle-[\mathbf{12}])(\langle\mathbf{13}\rangle-[\mathbf{13}])(\langle\mathbf{23}\rangle-[\mathbf{23}]). \tag{433}$$

(c) $L=1, S=1$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(1,1) = \mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2\}}\mathcal{Y}^1_{\{L_1L_2\}} C^{1,\{K_1K_2\};1,\{L_1L_2\}}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{434}$$

where $\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2\}}$ and $\mathcal{Y}^1_{\{L_1L_2\}}$ can be obtained from Eqs. (402) and (378), respectively.

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(1,1) = \sqrt{\frac{3}{2\pi|\mathbf{p}_{1\text{com}}|^3 m_3}}\frac{(m_1+m_2+m_3)^2(\langle\mathbf{12}\rangle-[\mathbf{12}])}{2(m_3(E_{1\text{com}}+m_1))(m_3(E_{2\text{com}}+m_2))} \times\Big((m_3+m_1-m_2)(\langle\mathbf{13}\rangle+[\mathbf{13}])(\langle\mathbf{23}\rangle-[\mathbf{23}]) + (m_1-m_3-m_2)(\langle\mathbf{13}\rangle-[\mathbf{13}])(\langle\mathbf{23}\rangle+[\mathbf{23}]). \tag{435}$$

(d) $L=2, S=1$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(2,1) = \mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2\}}\mathcal{Y}^2_{\{L_1L_2L_3L_4\}}C^{1,\{K_1K_2\};2,\{L_1L_2L_3L_4\}}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{436}$$

where $\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2\}}$ and $\mathcal{Y}^2_{\{L_1L_2L_3L_4\}}$ can be obtained from Eqs. (402) and (396), respectively.

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(2,1) = \sqrt{\frac{15}{8\pi|\mathbf{p}_{1\text{com}}|^5 m_3^3}}\frac{-(m_1+m_2+m_3)^2(m_3^2-(m_1-m_2)^2)}{(m_3(E_{1\text{com}}+m_1))(m_3(E_{2\text{com}}+m_2))}\sqrt{\frac{3}{5}} \times\Big(\frac{(m_1+m_2+m_3)}{4}(\langle\mathbf{12}\rangle-[\mathbf{12}])(\langle\mathbf{13}\rangle+[\mathbf{13}])(\langle\mathbf{23}\rangle+[\mathbf{23}]) + (\langle\mathbf{12}\rangle^2-[\mathbf{12}]^2)(\langle\mathbf{3}|p_1|\mathbf{3}]). \tag{437}$$

(e) $L=1, S=2$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(1,2) = \mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2K_3K_4\}}\mathcal{Y}^1_{\{L_1L_2\}}C^{2,\{K_1K_2K_3K_4\};1,\{L_1L_2\}}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{438}$$

where $\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2K_3K_4\}}$ and $\mathcal{Y}^1_{\{L_1L_2\}}$ can be obtained from Eqs. (405) and (378), respectively.

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(1,2) = \sqrt{\frac{3}{2\pi|\mathbf{p}_{1\text{com}}|^3 m_3}}\frac{(m_1+m_2+m_3)}{(m_3(E_{1\text{com}}+m_1))(m_3(E_{2\text{com}}+m_2))}\sqrt{\frac{3}{5}} \times\Big(((m_1-m_2)^2-m_3^2)(\langle\mathbf{13}\rangle-[\mathbf{13}])(\langle\mathbf{23}\rangle-[\mathbf{23}])(\langle\mathbf{12}\rangle+[\mathbf{12}]) + (m_2-m_3-m_1)\langle\mathbf{1}|p_3|\mathbf{1}](\langle\mathbf{23}\rangle-[\mathbf{23}])^2 + (m_3+m_2-m_1)\langle\mathbf{2}|p_3|\mathbf{2}](\langle\mathbf{13}\rangle-[\mathbf{13}])^2\Big). \tag{439}$$

(f) $L=2, S=2$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(2,2) = \mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2K_3K_4\}}\mathcal{Y}^2_{\{L_1L_2L_3L_4\}}C^{2,\{K_1K_2K_3K_4\};2,\{L_1L_2L_3L_4\}}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{440}$$

where $\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2K_3K_4\}}$ and $\mathcal{Y}^2_{\{L_1L_2L_3L_4\}}$ can be obtained from Eqs. (405) and (396), respectively.

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(2,2) = \sqrt{\frac{15}{8\pi|\mathbf{p}_{1\text{com}}|^5 m_3^3}}\frac{(m_1+m_2+m_3)^2(m_3^2-(m_1-m_2)^2)}{2(m_3(E_{1\text{com}}+m_1))(m_3(E_{2\text{com}}+m_2))}\sqrt{\frac{4}{5}} \times\Big((m_2-m_3-m_1)(\langle\mathbf{13}\rangle+[\mathbf{13}])(\langle\mathbf{23}\rangle-[\mathbf{23}])(\langle\mathbf{12}\rangle+[\mathbf{12}]) + (m_3+m_2-m_1)(\langle\mathbf{13}\rangle-[\mathbf{13}])(\langle\mathbf{23}\rangle+[\mathbf{23}])(\langle\mathbf{12}\rangle+[\mathbf{12}]) + \langle\mathbf{1}|p_3|\mathbf{1}](\langle\mathbf{23}\rangle^2-[\mathbf{23}]^2) + \langle\mathbf{2}|p_3|\mathbf{2}](\langle\mathbf{13}\rangle^2-[\mathbf{13}]^2)\Big). \tag{441}$$

(g) $L=3, S=2$:

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(3,2) = \mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2K_3K_4\}}\mathcal{Y}^3_{\{L_1L_2L_3L_4L_5L_6\}} \times C^{2,\{K_1K_2K_3K_4\};3,\{L_1L_2L_3L_4L_5L_6\}}_{1,\{I_1^{(3)}I_2^{(3)}\}}, \tag{442}$$

where $\mathcal{S}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{K_1K_2K_3K_4\}}$ can be obtained from Eq. (405), and

$$\mathcal{Y}^3_{\{L_1L_2L_3L_4L_5L_6\}} = \sqrt{\frac{35}{16\pi|\mathbf{p}_{1\text{com}}|^7 m_3^5}}(\langle\mathbf{3}|p_1|\mathbf{3}])^3. \tag{443}$$

The amplitude is

$$\mathcal{A}^{\{I_1^{(1)}I_2^{(1)}\},\{I_1^{(2)}I_2^{(2)}\}}_{\{I_1^{(3)}I_2^{(3)}\}}(3,2) = \sqrt{\frac{35}{16\pi|\mathbf{p}_{1\text{com}}|^7 m_3^5}}\frac{(m_1+m_2+m_3)^2(m_3^2-(m_1-m_2)^2)}{(m_3(E_{1\text{com}}+m_1))(m_3(E_{2\text{com}}+m_2))}\sqrt{\frac{3}{7}} \times\Big(\big(((m_3^2-(m_1-m_2)^2))(\langle\mathbf{12}\rangle+[\mathbf{12}])^2 - \langle\mathbf{1}|p_3|\mathbf{1}]\langle\mathbf{2}|p_3|\mathbf{3}]\big)\langle\mathbf{3}|p_1|\mathbf{3}] + \frac{(m_1+m_2+m_3)}{4}\big((m_1-m_3-m_2)(\langle\mathbf{23}\rangle+[\mathbf{23}])^2\langle\mathbf{1}|p_3|\mathbf{1}] + (m_3+m_1-m_2)(\langle\mathbf{13}\rangle+[\mathbf{13}])^2\langle\mathbf{2}|p_3|\mathbf{2}] + (m_3^2-(m_1-m_2)^2)(\langle\mathbf{12}\rangle+[\mathbf{12}])(\langle\mathbf{13}\rangle+[\mathbf{13}])(\langle\mathbf{23}\rangle+[\mathbf{23}])\big)\Big). \tag{444}$$

V. GENERAL ANALYSIS OF CASCADE DECAY

In the previous sections, both non-covariant and covariant methods have been introduced. In the non-covariant methods, two-body decay amplitudes are defined in the corresponding COM frames of the decays. Therefore, when the particle momenta are given in an arbitrary frame, such as the laboratory frame, it is necessary to boost the event to the relevant COM frames in order to evaluate the amplitudes. However, since such boosts are generally not along the momentum directions of the final-state particles, they often induce nontrivial Wigner rotations on the spin states of the particles.

If one were to directly sum or contract amplitudes defined in different COM frames, incorrect results would be obtained because the spin quantization axes in those frames are not aligned. It is therefore necessary to perform Wigner rotations to align the particle states, ensuring that the same particle is described with a consistent choice of quantization axis. Only after this alignment can different two-body decay amplitudes be summed or contracted consistently.

Because the definitions of the spin quantization axes differ in the canonical and helicity amplitudes, the treatment in cascade decays is also different. In the canonical amplitude, the spin quantization axis is chosen along the z-axis of a selected frame, whereas in the helicity amplitude, it is defined along the particle's momentum direction in the chosen frame. Consequently, the canonical and helicity formalisms require different procedures when applied to cascade decays.

In contrast, the covariant canonical-spinor two-body decay amplitudes are defined in an arbitrary frame and are constructed directly from the four-momenta in that frame. The spin quantization axes are all chosen to point along the z-axis of the selected frame, so the spin indices can be contracted and summed directly. Moreover, the momenta entering each amplitude are taken directly from the chosen frame, and no boost back to the COM frame is required. This makes the formalism particularly convenient when treating cascade decays with multiple decay chains.

In the following, we discuss separately the treatment of cascade decays in the non-covariant approach and in the covariant approach.

A. Non-covariant cascade decay

In this subsection, we primarily discuss the application of non-covariant approaches to cascade decays. Here, "non-covariant" means that the amplitude is not written in covariant form; specifically, it includes elements that must be evaluated by boosting back to the COM frame. In this context, the previously introduced non-covariant amplitudes include the helicity amplitude, the canonical amplitude, the covariant projection tensor amplitude in the PS scheme, and the Zemach tensor amplitude. These constructions are all defined starting from the COM frame at the level of their amplitude expressions. According to the definition of the spin quantization axes, these amplitudes can be further classified into amplitudes carrying canonical indices and amplitudes carrying helicity indices.

Therefore, in the canonical cascade decay, we discuss amplitudes with canonical indices, including the canonical amplitude, the covariant projection tensor amplitude in the PS scheme, and the Zemach amplitude. In the helicity cascade decay, we discuss the helicity amplitude.

1. *Canonical cascade decay*

In the previous sections, we presented several amplitude constructions that require boosting to the COM frame and ultimately carry canonical indices, including the **traditional-*LS* amplitude**, **the covariant projection tensor amplitude in the PS scheme**, and the **Zemach tensor amplitude**. In section 3, we showed that these three amplitudes are completely equivalent in the COM frame. Under a boost, all three amplitude constructions induce the same Wigner rotation acting on the final-state σ indices, and the rotation is identical for the three amplitudes. Therefore, in cascade decay analyses that require boosting back to the COM frame, they can be treated in the same way. In this subsubsection, we unify them under a single framework, which we refer to as the canonical cascade decay.

For a cascade decay, we first choose a global reference frame, such as the lab frame, and fix a set of axes (x,y,z) in that frame. In the lab frame, all external momenta p_i are expressed with respect to the same fixed coordinate system.

A key feature of the canonical basis is that the spin projection label σ_i is always defined as the projection onto this fixed lab frame z-axis. In other words, the spin polarization direction associated with every particle is tied to the same quantization axis. When each two-body decay amplitude is computed by boosting to the corresponding parent rest frame, the accompanying Wigner D-matrices precisely transport the daughter spin quantization axis back to this common quantization axis. As a result, all spin polarization directions appearing in different decay steps refer to the same spin quantization axis, and the intermediate-state spin sum is a well-defined contraction in a common quantization axis.

This has an immediate consequence for cascade decays and for processes with multiple decay chains leading to the same final state. Since every decay chain amplitude is ultimately expressed with external spin polarization directions defined with respect to the same lab frame z-axis, different chains can be added coherently without introducing any additional alignment rotations between different decay chains.

Next, we will give explicit examples for a single decay chain and for multiple decay chains, and provide a concrete method for calculating the cascade decay amplitude.

Canonical cascade decay amplitude for single decay chain As an explicit example, we consider the two-step cascade decay $X\to 3+Y$, followed by $3\to 1+2$. We define the lab frame to be the global frame associated with the fixed coordinate system above. In this lab frame, the momenta $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_Y$ of the final-state particles 1, 2, and Y are taken as the measured inputs. The four-momentum of the intermediate particle-3 is then reconstructed from momentum conservation as

$$\mathbf{p}_3=\mathbf{p}_1+\mathbf{p}_2,\qquad \mathbf{p}_X=\mathbf{p}_3+\mathbf{p}_Y. \tag{445}$$

According to the general expression for the canonical two-body decay amplitude in any frame, given in Eq. (96), the two two-body amplitudes entering the cascade $X\to 3+Y$ and $3\to 1+2$ can be written explicitly as follows:

1. Two-body decay $X\to 3+Y$ in the lab frame.

For the first step $X\to 3+Y$, boosting from the lab frame to the X rest frame by $L^{-1}_{X,c}(\mathbf{p}_X)$ gives the daughter momenta $\mathbf{p}^*_3$ and $\mathbf{p}^*_Y$. The corresponding amplitude in the lab frame is the two-body decay COM frame times Wigner D-matrices, which can be written as

$$\begin{aligned}&A^{\sigma_3\sigma_Y}_{\sigma_X}(\mathbf{p}_X,\mathbf{p}_3,\mathbf{p}_Y)\\&=\sum_{L_1S_1}\sum_{\sigma'_3\sigma'_Y}A^{\sigma'_3\sigma'_Y}_{\sigma_X}(\mathbf{k}_X,\mathbf{p}^*_3,\mathbf{p}^*_Y;L_1,S_1)D^{\sigma'_3(s_3)*}_{\sigma_3}(R_{X3})D^{\sigma'_Y(s_Y)*}_{\sigma_Y}(R_{XY}),\end{aligned} \tag{446}$$

where (L_1,S_1) denote the allowed LS combinations for the two-body decay. The Wigner rotations R_{X3} and R_{XY} are induced by the boost to the particle-X rest frame. Following Eq. (108), they can be written as

$$\begin{aligned}R_{X3}&=R_{\hat{\mathbf{n}}_{X3}}R_z(\omega_{X3})R^{-1}_{\hat{\mathbf{n}}_{X3}},\\R_{XY}&=R_{\hat{\mathbf{n}}_{XY}}R_z(\omega_{XY})R^{-1}_{\hat{\mathbf{n}}_{XY}},\end{aligned} \tag{447}$$

where the rotation axes are chosen as

$$\hat{\mathbf{n}}_{X3}\parallel -\mathbf{p}_X\times\mathbf{p}_3,\qquad \hat{\mathbf{n}}_{XY}\parallel -\mathbf{p}_X\times\mathbf{p}_Y. \tag{448}$$

The corresponding rotation angles about the z axis can be obtained from Eq. (107). Explicitly, for $i=3,Y$, one may write

$$\begin{aligned}&\tan\frac{\omega_{Xi}}{2}\\&=\frac{\sin\theta_{Xi}\sinh\left(\frac{\eta_X}{2}\right)\sinh\left(\frac{\eta_i}{2}\right)}{\cosh\left(\frac{\eta_X}{2}\right)\cosh\left(d\frac{\eta_i}{2}\right)+\cos\theta_{Xi}\sinh\left(\frac{\eta_X}{2}\right)\sinh\left(\frac{\eta_i}{2}\right)},\\&i=3,Y,\end{aligned} \tag{449}$$

where η_X is the rapidity of the canonical boost $L^{-1}_{X,c}(\mathbf{p}_X)$, η_i is the rapidity associated with the standard canonical boost $L_{i,c}(\mathbf{p}_i)$ of particle-i, and θ_{Xi} is the angle between $\mathbf{p}_X$ and $\mathbf{p}_i$ in the lab frame.

After the Wigner rotation, the amplitude in the lab frame has its quantization axis aligned with the lab z-axis.

2. Two-body decay $3\to 1+2$ in the lab frame.

For the second step $3\to 1+2$, boosting from the lab frame to the 3 rest frame by $L^{-1}_{3,c}(\mathbf{p}_3)$ gives the daughter momenta $\mathbf{p}^*_1$ and $\mathbf{p}^*_2$. The corresponding amplitude in the lab frame is the two-body decay COM frame times Wigner D-matrices, which can be written as

$$\begin{aligned}&A^{\sigma_1\sigma_2}_{\sigma_3}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2)\\&=\sum_{L_2S_2}\sum_{\sigma'_1\sigma'_2}A^{\sigma'_1\sigma'_2}_{\sigma_3}(\mathbf{k}_3,\mathbf{p}^*_1,\mathbf{p}^*_2;L_2,S_2)D^{\sigma'_1(s_1)*}_{\sigma_1}(R_{31})D^{\sigma'_2(s_2)*}_{\sigma_2}(R_{32}),\end{aligned} \tag{450}$$

where (L_2,S_2) denotes the allowed LS combinations for the two-body decay. The Wigner rotations R_{31} and R_{32} are induced by the boost to the particle-3 rest frame. Following Eq. (108), they can be written as

$$R_{31}=R_{\hat{\mathbf{n}}_{31}}R_z(\omega_{31})R^{-1}_{\hat{\mathbf{n}}_{31}},\qquad R_{32}=R_{\hat{\mathbf{n}}_{32}}R_z(\omega_{32})R^{-1}_{\hat{\mathbf{n}}_{32}}, \tag{451}$$

where the rotation axes are chosen as

$$\hat{\mathbf{n}}_{31}\parallel -\mathbf{p}_3\times\mathbf{p}_1,\qquad \hat{\mathbf{n}}_{32}\parallel -\mathbf{p}_3\times\mathbf{p}_2. \tag{452}$$

The corresponding rotation angles about the z-axis can be obtained from Eq. (107). Explicitly, for $i=1,2$, one may write

$$\begin{aligned}&\tan\frac{\omega_{3i}}{2}\\&=\frac{\sin\theta_{3i}\sinh\left(\frac{\eta_3}{2}\right)\sinh\left(\frac{\eta_i}{2}\right)}{\cosh\left(\frac{\eta_3}{2}\right)\cosh\left(\frac{\eta_i}{2}\right)+\cos\theta_{3i}\sinh\left(\frac{\eta_3}{2}\right)\sinh\left(\frac{\eta_i}{2}\right)},\\&i=1,2,\end{aligned} \tag{453}$$

where η_3 is the rapidity of the canonical boost $L^{-1}_{3,c}(\mathbf{p}_3)$, η_i

is the rapidity associated with the standard canonical boost $L_{i,c}(\mathbf{p}_i)$ of particle-i, and θ_{3i} is the angle between $\mathbf{p}_3$ and $\mathbf{p}_i$ in the lab frame.

After the Wigner rotation, the amplitude in the lab frame has its quantization axis aligned with the lab z-axis.

In each of the two-body decays mentioned above, we have aligned the spin quantization axes of the particles to the lab frame's z-axis via Wigner rotations. Therefore, in both two-body decay amplitudes, the spin quantum number σ_3 of particle-3 is always defined relative to the same direction, the lab frame's z-axis. With a sum over the canonical spin projection of the intermediate particle-3, the total cascade decay amplitude can be written as

$$A_{\sigma_X}^{\sigma_1\sigma_2\sigma_Y}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y) = \sum_{\sigma_3} A_{\sigma_X}^{\sigma_3\sigma_Y}(\mathbf{p}_X,\mathbf{p}_3,\mathbf{p}_Y)P_3 A_{\sigma_3}^{\sigma_1\sigma_2}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2), \tag{454}$$

where P_3 is the resonant propagator of the intermediate particle-3, the amplitude for the cascade decay also has its quantization axis aligned with the same direction, namely the lab frame z-axis.

The geometry of the cascade decay process is shown in Fig. 7.

Canonical cascade decay amplitude for multiple decay chains For a cascade process with multiple decay chains, each chain is defined in the same lab frame. Since the canonical spin projections of all final-state particles are defined with respect to the same fixed lab-frame z-axis, the spin polarization directions are consistent among different chains. Therefore, one can first compute the cascade amplitude of each decay chain in the lab frame, and then obtain the total amplitude by directly summing the chain amplitudes.

As an explicit example, we consider two decay chains,

- Decay chain 1: $X\to 3+Y$, followed by $3\to 1+2$.
- Decay chain 2: $X\to 4+2$, followed by $4\to 1+Y$.

In the lab frame, following the single chain construction discussed above and using Eq. (454), the corresponding cascade decay amplitudes are given by the following expressions:

$$\begin{aligned} &A_{\sigma_X}^{\sigma_1\sigma_2\sigma_Y}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y;3)\\ &= \sum_{\sigma_3} A_{\sigma_X}^{\sigma_3\sigma_Y}(\mathbf{p}_X,\mathbf{p}_3,\mathbf{p}_Y)P_3 A_{\sigma_3}^{\sigma_1\sigma_2}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2),\\ &A_{\sigma_X}^{\sigma_1\sigma_2\sigma_Y}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y;4)\\ &= \sum_{\sigma_4} A_{\sigma_X}^{\sigma_4\sigma_2}(\mathbf{p}_X,\mathbf{p}_4,\mathbf{p}_2)P_4 A_{\sigma_4}^{\sigma_1\sigma_Y}(\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_Y). \end{aligned} \tag{455}$$

The total cascade amplitude for this process is obtained by directly summing the two chain amplitudes as

$$A_{\sigma_X}^{\sigma_1\sigma_2\sigma_Y}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y) = A_{\sigma_X}^{\sigma_1\sigma_2\sigma_Y}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y;3)+A_{\sigma_X}^{\sigma_1\sigma_2\sigma_Y}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y;4). \tag{456}$$

Finally, an advantage of the canonical cascade construction is that the spin quantization axis of the particles is consistent for each decay chain. This allows the total cascade amplitude to be obtained by directly summing the cascade amplitudes of all decay chains in the lab frame.

However, in practical calculations, evaluating each two-body decay amplitude in any frame still requires computing the two-body amplitude in the corresponding COM frame, along with the Wigner rotations. Once multiple decay chains are involved, the computational cost becomes substantial, which is not very convenient in practice.

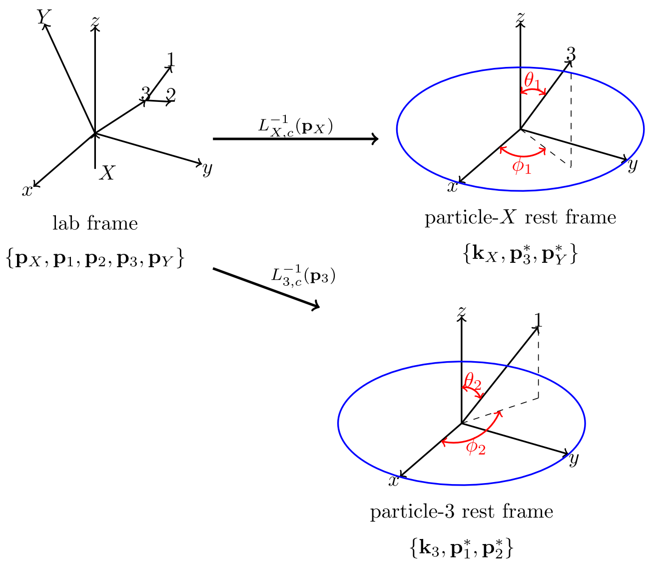

Fig. 7. (color online) $L^{-1}_{X,c}(\mathbf{p}_X)$ is the canonical-standard boost of particle-X from the lab frame to the rest frame of particle-X, where the momenta are $\{\mathbf{k}_X,\mathbf{p}^*_3,\mathbf{p}^*_Y\}$. $L^{-1}_{3,c}(\mathbf{p}_3)$ is the canonical-standard boost of particle-3 from the lab frame to the rest frame of particle-3, where the momenta are $\{\mathbf{k}_3,\mathbf{p}^*_1,\mathbf{p}^*_2\}$. The angles shown correspond to those used in the amplitudes defined above.

2. *Helicity cascade decay*

In this subsubsection, we discuss the treatment of helicity amplitudes in cascade decays. Since helicity is defined with respect to the momentum direction in a given reference frame, different reference frames generally lead to different helicity definitions. In the following, we focus on the definitions of cascade decay amplitudes under two different helicity conventions.

● **Two-body decay COM frame helicity (local helicity).** For each two-body decay $a \to b+c$, one defines the helicities $\lambda_b^{(a)}$ and $\lambda_c^{(a)}$ in the rest frame of particle-a. This choice yields the standard Jacob-Wick factorization of each two-body amplitude into the Wigner D-function of the decay angles and the helicity-coupling amplitude.

In most of the literature, this definition is used to construct cascade decay amplitudes. When the cascade decay involves only a single decay chain, it is equivalent to the lab frame description and yields the correct results for physical observables. However, once multiple decay chains are present, one cannot directly add the amplitudes from different chains. This is because different decay chains correspond to different topologies, and in general, the final-state helicities associated with different chains are not defined in the same way. A naive direct sum would, therefore, lead to incorrect results.

The correct procedure is to choose one reference decay chain and then rotate the final-state helicities of the other chains so that they are aligned with those of the reference chain, thereby obtaining cascade amplitudes defined with the same final-state helicity convention []. After this alignment rotation, the cascade amplitudes from different decay chains can be added directly, and the total cascade decay amplitude is obtained by summing over all chains.

● **Lab helicity (global helicity).** For the final-state particle-i, one may define the helicity λ_i in a fixed global frame, for example, the lab frame, by projecting the spin onto the momentum direction in the lab frame.

Under this convention, each two-body decay amplitude is defined in the lab frame, and the helicities of all particles are defined along their respective momentum directions in the lab frame. For intermediate particles, this guarantees that the spin quantization axis is defined in the same way when they appear as daughter particles and as parent particles. Therefore, for a given cascade decay chain, one can directly sum over the helicity of the intermediate state and obtain the amplitude for that chain.

Moreover, when multiple decay chains are present, the final-state helicities in every chain are defined in the same lab frame. The spin quantization axes for the same final-state particle in different chains are always taken along that particle's lab frame momentum direction. As a consequence, there is no mismatch in final-state helicity conventions across different chains, and one can directly add the cascade decay amplitudes from different chains to obtain the total cascade decay amplitude.

Based on the two helicity conventions introduced above, we first use the case of a single decay chain to illustrate how the corresponding cascade decay amplitude is constructed under each convention. We then turn to the situation with multiple decay chains and explain how the cascade decay amplitude with several chains is constructed in each convention.

Helicity cascade decay amplitude for single decay chain Consider a cascade decay process $X \to 3+Y$, followed by $3 \to 1+2$, where all particles are massive and carry spin. According to the two helicity conventions introduced above, the corresponding cascade amplitude can be constructed in two different ways.

For the two-body decay COM frame helicity convention, we directly construct the cascade decay amplitude from the two-body amplitudes defined in the corresponding COM frames, and then obtain the total cascade amplitude as

$$\begin{aligned}&A_{\lambda_X}^{\lambda_1\lambda_2\lambda_Y}(\mathbf{k}_X,\mathbf{q}_Y^*,\mathbf{q}_1^*,\mathbf{q}_2^*)\\&=\sum_{\lambda_3}A_{\lambda_X}^{\lambda_3\lambda_Y}(\mathbf{k}_X,\mathbf{q}_3^*,\mathbf{q}_Y^*)P_3A_{\lambda_3}^{\lambda_1\lambda_2}(\mathbf{k}_3,\mathbf{q}_1^*,\mathbf{q}_2^*),\end{aligned} \tag{457}$$

where each helicity λ_i is defined along the momentum direction in the COM frame of the corresponding two-body decay.

In contrast, for the lab frame helicity convention, we construct the cascade decay amplitude using the two-body decay amplitudes written in the lab frame, and then obtain the total cascade amplitude as

$$\begin{aligned}&A_{\lambda_X}^{\lambda_1'\lambda_2'\lambda_Y'}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y)\\&=\sum_{\lambda_3'}A_{\lambda_X}^{\lambda_3'\lambda_Y'}(\mathbf{p}_X,\mathbf{p}_3,\mathbf{p}_Y)P_3A_{\lambda_3'}^{\lambda_1'\lambda_2'}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2),\end{aligned} \tag{458}$$

where each helicity λ_i' is defined along the momentum direction in the lab frame. In this cascade decay, the helicity of the parent particle-X is the same in both conventions.

1. Two-body decay COM frame helicity convention

$$\begin{aligned}&A_{\lambda_X}^{\lambda_1\lambda_2\lambda_Y}(\mathbf{k}_X,\mathbf{q}_Y^*,\mathbf{q}_1^*,\mathbf{q}_2^*)\\&=\sum_{\lambda_3}A_{\lambda_X}^{\lambda_3\lambda_Y}(\mathbf{k}_X,\mathbf{q}_3^*,\mathbf{q}_Y^*)P_3A_{\lambda_3}^{\lambda_1\lambda_2}(\mathbf{k}_3,\mathbf{q}_1^*,\mathbf{q}_2^*),\end{aligned} \tag{459}$$

where each amplitude is defined in the helicity COM frame. The helicity COM frame is determined by starting from the initial COM frame and applying the helicity-standard boost to an intermediate particle produced from the two-body decay of the parent particle, bringing it to its rest frame. This frame is then defined as the COM frame for the subsequent two-body decay of that intermediate particle as the new parent. Proceeding step by step via successive helicity-standard boosts in this manner, we obtain the helicity COM frames for all two-body decays in the cascade.

Here, we can sum over the helicities of the intermediate state particle in the two amplitudes because, for the intermediate particle, the helicity quantization axes are consistent between the two helicity COM frames. Suppose we consider the first two-body decay, where the particle state is $|\mathbf{q}_3^*,\lambda_3\rangle$, and in the second two-body decay, the particle state is $|\mathbf{k}_3,\lambda_3'\rangle$. The transformation between these two helicity COM frames is given by the helicity-standard boost for particle-3 with momentum $\mathbf{q}_3^*$, which can be decomposed as

$$L_{3,h}^{-1}(\mathbf{q}_3^*) = R^{-1}(\phi_1,\theta_1,0)L_{3,c}^{-1}(\mathbf{q}_3^*), \tag{460}$$

where $L_{3,c}^{-1}(\mathbf{q}_3^*)$ is a pure boost along the momentum direction $\mathbf{q}_3^*$. Due to the rotational invariance of helicity and its invariance under pure boosts along the momentum direction, we ultimately have

$$\begin{aligned} L_{3,h}^{-1}(\mathbf{q}_3^*)|\mathbf{q}_3^*,\lambda_3\rangle &= R^{-1}(\phi_1,\theta_1,0)L_{3,c}^{-1}(\mathbf{q}_3^*)|\mathbf{q}_3^*,\lambda_3\rangle \\ &= |\mathbf{k}_3,\lambda_3\rangle, \end{aligned} \tag{461}$$

which indicates that $\lambda_3 = \lambda_3'$. Thus, the choice of helicity quantization axis for the intermediate particle state is consistent, and we can directly sum over them. This demonstrates that when defining the helicity COM frame, it must be obtained by boosting from the decay COM frame of its parent particle. This ensures that the helicity quantization axes for each intermediate particle are consistent, thus allowing summation over them.

Starting from the first step in the two-body decay $X\to 3+Y$, in the particle-X rest frame, from Eq. (75), the helicity amplitude in the COM frame can be written as

$$A_{\lambda_X}^{\lambda_3\lambda_Y}(\mathbf{k}_X,\mathbf{q}_3^*,\mathbf{q}_Y^*) = \sqrt{\frac{2s_X+1}{4\pi}}D_{\lambda_3-\lambda_Y}^{\lambda_X(s_X)*}(\phi_1,\theta_1,0)H^{\lambda_3\lambda_Y}, \tag{462}$$

where (θ_1,ϕ_1) are the corresponding helicity angles.

For the second step two-body decay $3\to 1+2$, in the particle-3 rest frame, the helicity amplitude can be written as

$$A_{\lambda_3}^{\lambda_1\lambda_2}(\mathbf{k}_3,\mathbf{q}_1^*,\mathbf{q}_2^*) = \sqrt{\frac{2s_3+1}{4\pi}}D_{\lambda_1-\lambda_2}^{\lambda_3(s_3)*}(\phi_2,\theta_2,0)H^{\lambda_1\lambda_2}, \tag{463}$$

where (θ_2,ϕ_2) are the corresponding helicity angles. The geometric definitions of these two sets of angles can be found in Fig. 8.

2. Lab helicity convention

Under the lab frame convention, the helicities of all particles are defined with respect to the directions of their momenta in the lab frame. Therefore, one can compute each two-body subprocess in the cascade decay by using the lab frame helicity two-body decay formula, Eq. (115). In particular, for the intermediate resonance, its helicity is defined along its lab momentum direction both when it appears as a daughter particle and when it appears as a parent particle. Hence, the helicity of the intermediate state is the same in the two roles, and the intermediate helicity can be summed over directly. As a result, the cascade decay amplitude can be written as

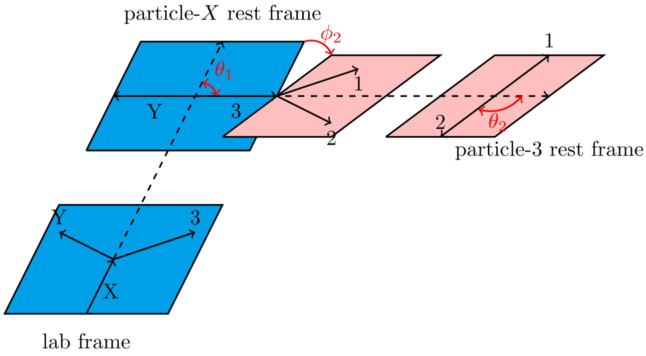

Fig. 8. (color online) Definitions of helicity angles for the decay chain $X\to 3+Y$, $3\to 1+2$. For the first two-body decay $X\to 3+Y$, we may choose the decay plane such that $\phi_1 = 0$. For a single decay chain, this only introduces an overall phase and does not affect physical results. The corresponding helicity angles for this decay are therefore $(\theta_1,0)$. For the second two-body decay, the rest frame of particle-3 is obtained from the rest frame of particle-X by applying the helicity-standard boost of particle-3, $L_{3,h}^{-1}(\mathbf{q}_3^*)$. In this frame, the helicity angles of the decay $3\to 1+2$ are (θ_2,ϕ_2).

$$\begin{aligned} &A_{\lambda_X}^{\lambda_1'\lambda_2'\lambda_Y'}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y) \\ &= \sum_{\lambda_3'} A_{\lambda_X}^{\lambda_3'\lambda_Y'}(\mathbf{p}_X,\mathbf{p}_3,\mathbf{p}_Y)P_3A_{\lambda_3'}^{\lambda_1'\lambda_2'}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2). \end{aligned} \tag{464}$$

For the two-body decay $X\to 3+Y$, the two-body decay amplitude in the lab frame can be expressed as

$$\begin{aligned} &A_{\lambda_X}^{\lambda_3'\lambda_Y'}(\mathbf{p}_X,\mathbf{p}_3,\mathbf{p}_Y) \\ &= \sum_{\tilde\lambda_3\tilde\lambda_Y} A_{\lambda_X}^{\tilde\lambda_3\tilde\lambda_Y}(\mathbf{k}_X,\tilde{\mathbf{q}}_3^*,\tilde{\mathbf{q}}_Y^*)D_{\lambda_3'}^{\tilde\lambda_3(s_3)*}(R_{X3}')D_{\lambda_Y'}^{\tilde\lambda_Y(s_Y)*}(R_{XY}'), \end{aligned} \tag{465}$$

where the helicity COM frame associated with $A_{\lambda_X}^{\tilde\lambda_3\tilde\lambda_Y}(\mathbf{k}_X,\tilde{\mathbf{q}}_3^*,\tilde{\mathbf{q}}_Y^*)$ is obtained from the lab frame by applying the helicity-standard boost of particle-X. This choice ensures that in the rest frame of particle-X, its spin quantization axis is the z-axis. When boosting the system $X\to 3+Y$, for the state of particle-3, helicity is invariant under rotations and also invariant under boosts along its momentum direction, so its helicity coincides with the helicity defined in the lab frame. For particle-3 and particle-Y, the helicity boost induces the Wigner rotation R'_{Xi} $(i=3,Y)$. This is done to ensure that, in the final two-body decay amplitude written in the lab frame, the helicities of particle-3 and particle-Y are defined along their momentum directions in the lab frame. From Eq. (122), this Wigner rotation can be written as

$$R'_{Xi} = R^{-1}(\tilde{\mathbf{q}}_i^*)R^{-1}(\mathbf{p}_X)R_{Xi}(\omega_i)R(\mathbf{p}_X), \tag{466}$$

where

$$\begin{aligned}
&R_{Xi}(\omega_i) = R_{\hat{\mathbf{n}}}R_z(\omega_{Xi})R_{\hat{\mathbf{n}}}^{-1},\\
&\tan\frac{\omega_{Xi}}{2} = \frac{\sin\theta_{Xi}\sinh\left(\frac{\eta_X}{2}\right)\sinh\left(\frac{\eta_i}{2}\right)}{\cosh\left(\frac{\eta_X}{2}\right)\cosh\left(\frac{\eta_i}{2}\right)+\cos\theta_{Xi}\sinh\left(\frac{\eta_X}{2}\right)\sinh\left(\frac{\eta_i}{2}\right)}\\
&\times\hat{\mathbf{n}}_{Xi}\parallel -\mathbf{p}_X\times\mathbf{p}_i.
\end{aligned} \tag{467}$$

The helicity amplitude in the lab frame can be expressed as

$$\begin{aligned}
&A_{\lambda_X}^{\lambda'_3\lambda'_Y}(\mathbf{p}_X,\mathbf{p}_3,\mathbf{p}_Y)\\
&=\sum_{\tilde{\lambda}_3\tilde{\lambda}_Y}\sqrt{\frac{2s_X+1}{4\pi}}D^{\lambda_X(s_X)*}_{\tilde{\lambda}_3-\tilde{\lambda}_Y}(\phi'_1,\theta'_1,0)H^{\tilde{\lambda}_3\tilde{\lambda}_Y}\\
&\quad\times D^{\tilde{\lambda}_3(s_3)*}_{\lambda'_3}(R'_{X3})D^{\tilde{\lambda}_Y(s_Y)*}_{\lambda'_Y}(R'_{XY}).
\end{aligned} \tag{468}$$

For the two-body decay $3\to1+2$, the approach here differs from the two-body decay COM frame helicity convention. In the two-body decay COM frame helicity convention, one must perform step-by-step helicity boosts to reach the corresponding helicity COM frame. However, here it is sufficient to start from the lab frame and apply a single helicity standard boost associated with the lab frame momentum of particle-3, bringing particle-3 to its rest frame. In this case, the two-body decay amplitude in the lab frame can be written as

$$A_{\lambda'_3}^{\lambda'_1\lambda'_2}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2)=\sum_{\tilde{\lambda}_1\tilde{\lambda}_2}A_{\lambda'_3}^{\tilde{\lambda}_1\tilde{\lambda}_2}(\mathbf{k}_3,\tilde{\mathbf{q}}_1^*,\tilde{\mathbf{q}}_2^*)D^{\tilde{\lambda}_1(s_1)*}_{\lambda'_1}(R'_{31})D^{\tilde{\lambda}_2(s_2)*}_{\lambda'_2}(R'_{32}). \tag{469}$$

For particle-3, this helicity boost consists of a rotation followed by a pure boost along the momentum direction of particle-3. Therefore, the helicity of particle-3 in the two-body COM frame coincides with its helicity defined in the lab frame. For particle-1 and particle-2, one applies the Wigner rotation R_{3i} $(i=1,2)$ induced by the helicity boost to ensure that in the final two-body decay amplitude written in the lab frame, the helicities of particle-1 and particle-2 are defined along their momentum directions in the lab frame. From Eq. (122), this Wigner rotation can be written as

$$R'_{3i} = R^{-1}(\tilde{\mathbf{q}}_i^*)R^{-1}(\mathbf{p}_3)R_{3i}(\omega_i)R(\mathbf{p}_3), \tag{470}$$

where

$$\begin{aligned}
&R_{3i}(\omega_i) = R_{\hat{\mathbf{n}}}R_z(\omega_{3i})R_{\hat{\mathbf{n}}}^{-1},\\
&\tan\frac{\omega_{3i}}{2} = \frac{\sin\theta_{3i}\sinh\left(\frac{\eta_3}{2}\right)\sinh\left(\frac{\eta_i}{2}\right)}{\cosh\left(\frac{\eta_3}{2}\right)\cosh\left(\frac{\eta_i}{2}\right)+\cos\theta_{3i}\sinh\left(\frac{\eta_3}{2}\right)\sinh\left(\frac{\eta_i}{2}\right)},\\
&\hat{\mathbf{n}}_{3i}\parallel -\mathbf{p}_3\times\mathbf{p}_i.
\end{aligned} \tag{471}$$

The helicity amplitude in the lab frame can be expressed as

$$\begin{aligned}
A_{\lambda'_3}^{\lambda'_1\lambda'_2}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2) = &\sum_{\tilde{\lambda}_1\tilde{\lambda}_2}\sqrt{\frac{2s_3+1}{4\pi}}D^{\lambda'_3(s_3)*}_{\tilde{\lambda}_1-\tilde{\lambda}_2}(\phi'_2,\theta'_2,0)H^{\tilde{\lambda}_1\tilde{\lambda}_2}\\
&\times D^{\tilde{\lambda}_1(s_1)*}_{\lambda'_1}(R'_{31})D^{\tilde{\lambda}_2(s_2)*}_{\lambda'_2}(R'_{32}).
\end{aligned} \tag{472}$$

In each of the two-body decays discussed above, we employ Wigner rotations to align the spin quantization axes with the momentum directions of the corresponding particles in the lab frame. Consequently, in the amplitudes of these two-body decays, the helicity of particle-3 is consistently defined along the momentum direction of particle-3 in the lab frame. By summing over the helicity of the intermediate particle-3, the total cascade decay amplitude can be expressed as

$$\begin{aligned}
&A_{\lambda_X}^{\lambda'_1\lambda'_2\lambda'_Y}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y)\\
&=\sum_{\lambda'_3}A_{\lambda_X}^{\lambda'_3\lambda'_Y}(\mathbf{p}_X,\mathbf{p}_3,\mathbf{p}_Y)P_3A_{\lambda'_3}^{\lambda'_1\lambda'_2}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2).
\end{aligned} \tag{473}$$

The geometry of the cascade decay process is shown in Fig. 9.

The two constructions above differ only by their choice of helicity convention. They are, therefore, physically equivalent, provided that the corresponding Wigner D-matrices induced by non-collinear boosts are treated consistently. In particular, when we are concerned with some physical observables, we sum over the final-state helicities and take the modulus squared of the amplitude. Thus, one has

$$\sum_{\lambda'_1\lambda'_2\lambda'_Y\lambda_X}\left|A_{\lambda_X}^{\lambda'_1\lambda'_2\lambda'_Y}\right|^2=\sum_{\lambda_1\lambda_2\lambda_Y\lambda_X}\left|A_{\lambda_X}^{\lambda_1\lambda_2\lambda_Y}\right|^2, \tag{474}$$

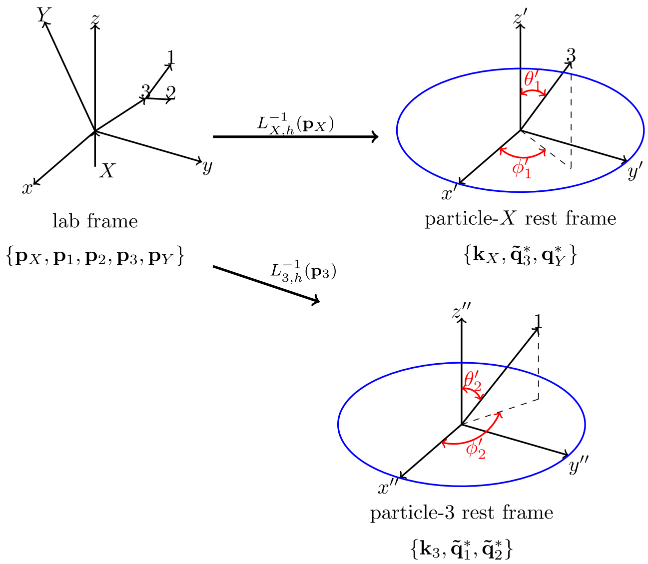

Fig. 9. (color online) $L^{-1}_{X,h}(\mathbf{p}_X)$ is the helicity boost from the lab frame to the rest frame of particle-X. It ensures that after the boost, the spin quantization axis of particle-X points along the z' direction, with the associated coordinate system denoted by $\{x',y',z'\}$. In the two-body decay $X\to 3+Y$ COM frame, the momenta of the particles in this frame are $\mathbf{k}_X, \tilde{\mathbf{q}}^*_3, \tilde{\mathbf{q}}^*_Y$, and the helicity angles are $(\phi'_1,\theta'_1,0)$. $L^{-1}_{3,h}(\mathbf{p}_3)$ is the helicity boost from the lab frame to the rest frame of particle-3. It ensures that after the boost, the spin quantization axis of particle-3 points along the z'' direction, with the associated coordinate system denoted by $\{x'',y'',z''\}$. In the two-body decay $3\to 1+2$ COM frame, the momenta of the particles in this frame are $\mathbf{k}_3, \tilde{\mathbf{q}}^*_1, \tilde{\mathbf{q}}^*_2$, and the helicity angles are $(\phi'_2,\theta'_2,0)$.

so both conventions lead to the same physical decay rate.

Despite this equivalence for a single decay chain, the situation becomes practically different when multiple decay chains contribute to the same final state. In the two-body COM frame helicity convention, the step-by-step helicity constructions are performed in different two-body rest frames, and different decay topologies generally lead to different helicity definitions (spin quantization axes) for the same external particle. As a result, there is a basis mismatch between chain amplitudes, and a direct coherent sum of the amplitudes from different chains is generally incorrect. To obtain a meaningful total amplitude, one must first bring all chain amplitudes into a common helicity basis by applying the appropriate alignment rotations.

In contrast, in the lab-frame helicity convention, the helicities of all external particles are defined with respect to their momentum directions in the same lab frame. Therefore, the spin bases are already aligned across different decay chains, and the chain amplitudes can be coherently summed directly without any additional alignment. In the following, we discuss these two treatments in detail.

Helicity cascade decay amplitude for multiple decay chains

1. Two-body decay COM frame helicity convention for multiple decay chains

According to the convention of the two-body decay helicity COM frame, since the topological structure of each cascade decay chain differs, the step-by-step helicity boost procedure from the initial COM frame to the frame in which the final-state particles reside must differ. Therefore, when summing the amplitudes from different decay chains, a mismatch in the definition of the final-state particle helicities arises. To resolve this issue and correctly sum the contributions from each decay chain, we must introduce the Wigner rotation to align the helicities of the final-state particles. Upon alignment, the amplitudes from each decay chain can be coherently summed to obtain the total cascade decay amplitude. Next, we will discuss in detail the implementation of this alignment rotation.

As an explicit example, we consider two decay chains,

- Decay chain-1: $X\to 3+Y$, followed by $3\to 1+2$.

- Decay chain-2: $X\to 4+2$, followed by $4\to 1+Y$.

The helicity amplitude for decay chain-1 is

$$\begin{aligned} &A^{\lambda_1\lambda_2\lambda_Y}_{\lambda_X}(\mathbf{k}_X,\mathbf{q}^*_Y,\mathbf{q}^*_1,\mathbf{q}^*_2;3)\\ &=\sum_{\lambda_3} A^{\lambda_3\lambda_Y}_{\lambda_X}(\mathbf{k}_X,\mathbf{q}^*_3,\mathbf{q}^*_Y)P_3 A^{\lambda_1\lambda_2}_{\lambda_3}(\mathbf{k}_3,\mathbf{q}^*_1,\mathbf{q}^*_2). \end{aligned} \tag{475}$$

For the two-body decay $X\to 3+Y$, the helicity of particle-X (λ_X) is defined along the z-axis in the COM frame of this two-body decay, while the helicities of particle-3 and particle-Y (λ_3,λ_Y) are defined along their respective momentum directions in the same COM frame.

For the two-body decay $3\to 1+2$, the helicity of particle-3 (λ_3) is defined along the z'-axis in the COM frame of this two-body decay (this z'-axis is determined by applying the helicity boost from the COM frame of the decay $X\to 3+Y$), while the helicities of particle-1 and particle-2 (λ_1,λ_2) are defined along their respective momentum directions in that COM frame.

As for the helicity of particle-3 in these two frames, since helicity is invariant under rotations and invariant under boosts along the direction of momentum, and because the transformation from the rest frame of X to the rest frame of particle-3 is achieved by a single helicity boost, the helicity of particle-3 λ_3 remains unchanged between the two rest frames. Equivalently, the spin quantization axis associated with particle-3 is the same in both decays. Therefore, one can sum over the intermediate helicity of particle-3 to obtain the cascade decay amplitude. The amplitudes in these two COM frames can be written as

$$A_{\lambda_X}^{\lambda_3\lambda_Y}(\mathbf{k}_X,\mathbf{q}_3^*,\mathbf{q}_Y^*)=\sqrt{\frac{2s_X+1}{4\pi}}D_{\lambda_3-\lambda_Y}^{\lambda_X(s_X)*}(\phi_1,\theta_1,0)H^{\lambda_3\lambda_Y},\quad(476)$$

and

$$A_{\lambda_3}^{\lambda_1\lambda_2}(\mathbf{k}_3,\mathbf{q}_1^*,\mathbf{q}_2^*)=\sqrt{\frac{2s_3+1}{4\pi}}D_{\lambda_1-\lambda_2}^{\lambda_3(s_3)*}(\phi_2,\theta_2,0)H^{\lambda_1\lambda_2},\quad(477)$$

where (θ_1,ϕ_1) and (θ_2,ϕ_2) are the corresponding helicity angles shown in Fig. 10.

The helicity amplitude for the decay chain-2 is

$$\begin{aligned}&A_{\lambda_X}^{\bar\lambda_1\bar\lambda_2\bar\lambda_Y}(\mathbf{k}_X,\bar{\mathbf{q}}_Y^*,\bar{\mathbf{q}}_1^*,\bar{\mathbf{q}}_2^*;4)\\&=\sum_{\bar\lambda_4}A_{\lambda_X}^{\bar\lambda_4\bar\lambda_2}(\mathbf{k}_X,\bar{\mathbf{q}}_4^*,\bar{\mathbf{q}}_2^*)P_4A_{\bar\lambda_4}^{\bar\lambda_1\bar\lambda_Y}(\mathbf{k}_4,\bar{\mathbf{q}}_1^*,\bar{\mathbf{q}}_Y^*).\end{aligned}\quad(478)$$

For the two-body decay $X\to4+2$, the helicity of particle-X (λ_X) is defined along the z-axis in the COM frame of this two-body decay, while the helicities of particle-4 and particle-2 ($\bar\lambda_4,\bar\lambda_2$) are defined along their respective momentum directions in the same COM frame.

For the two-body decay $4\to1+Y$, the helicity of particle-4 ($\bar\lambda_4$) is defined along the z''-axis in the COM frame of this two-body decay. This z''-axis is fixed by applying the helicity boost from the COM frame of the decay $X\to4+2$, while the helicities of particle-1 and particle-Y ($\bar\lambda_1,\bar\lambda_Y$) are defined along their respective momentum directions in that COM frame.

The helicity of particle-4 remains unchanged between the two COM frames. Equivalently, the spin quantization axis associated with particle-4 is the same in both decays. Therefore, one can sum over the intermediate helicity of particle-4 to obtain the cascade decay amplitude. The amplitudes in these two COM frames can be written as

$$A_{\lambda_X}^{\bar\lambda_4\bar\lambda_2}(\mathbf{k}_X,\bar{\mathbf{q}}_4^*,\bar{\mathbf{q}}_2^*)=\sqrt{\frac{2s_X+1}{4\pi}}D_{\bar\lambda_4-\bar\lambda_2}^{\lambda_X(s_X)*}(\bar\phi_1,\bar\theta_1,0)H^{\bar\lambda_4\bar\lambda_2},\quad(479)$$

and

$$A_{\bar\lambda_4}^{\bar\lambda_1\bar\lambda_Y}(\mathbf{k}_4,\bar{\mathbf{q}}_1^*,\bar{\mathbf{q}}_Y^*)=\sqrt{\frac{2s_4+1}{4\pi}}D_{\bar\lambda_1-\bar\lambda_Y}^{\bar\lambda_4(s_4)*}(\bar\phi_2,\bar\theta_2,0)H^{\bar\lambda_1\bar\lambda_Y},\quad(480)$$

where $(\bar\theta_1,\bar\phi_1)$ and $(\bar\theta_2,\bar\phi_2)$ are the corresponding helicity angles shown in Fig. 11.

One can see that the two cascade decay amplitudes define the final-state helicities with different directions; namely, they correspond to different spin quantization axes. To obtain a common set of quantization axes, we next align one decay chain to the reference chain. Here, we choose decay chain-1 as the reference, and decay chain-2 is to be aligned with it.

We first define the relations among the relevant reference frames in decay chain-1.

- The decay $X\to3+Y$ is defined in the rest frame of particle-X, which we denote by $\boldsymbol{S}_X$.

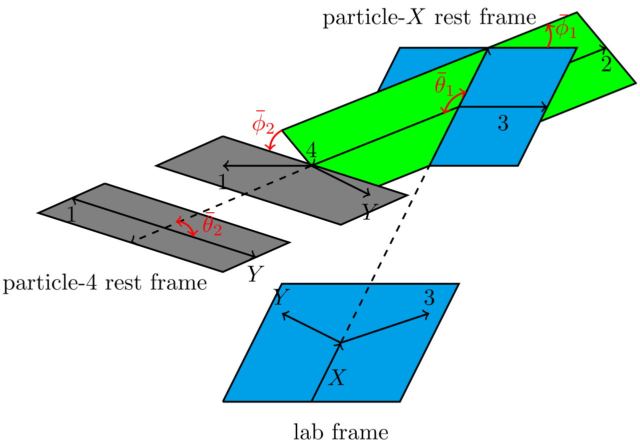

Fig. 10. (color online) Definitions of helicity angles for the first decay chain $X\to3+Y$, $3\to1+2$. This decay chain is taken as the reference chain. We define the rest frame of particle-X as the coordinate system associated with the blue plane. For the first two-body decay $X\to3+Y$, the helicity angles are $(\theta_1,0)$. For the second two-body decay $3\to1+2$, defined in the rest frame of particle-3, the helicity angles are (θ_2,ϕ_2).

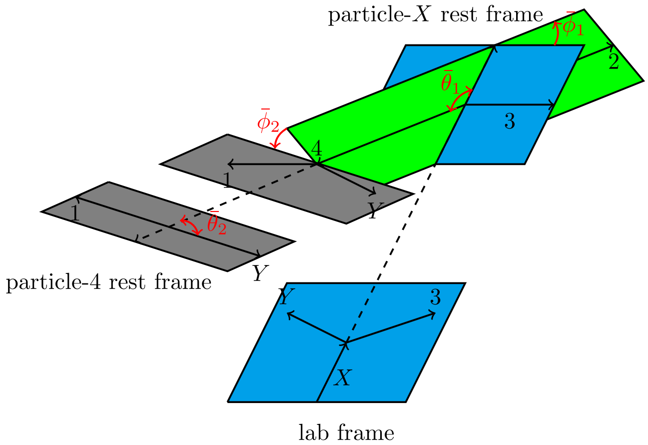

Fig. 11. (color online) Definitions of helicity angles for the second decay chain $X\to2+4$, $4\to1+Y$. The rest frame of particle-X is taken to be the coordinate system associated with the blue plane. For the first two-body decay $X\to2+4$, the helicity angles are $(\bar\theta_1,\bar\phi_1)$. For the second two-body decay $4\to1+Y$, defined in the rest frame of particle-4, the helicity angles are $(\bar\theta_2,\bar\phi_2)$.

● In the rest frame of particle-Y, obtained by applying the helicity boost from the rest frame of particle-X, we define this frame as $\boldsymbol{S}_Y$.

● The second two-body decay $3 \to 1+2$ is defined in the rest frame of particle-3, which we denote by $\boldsymbol{S}_3$.

● By boosting from the rest frame of particle-3 to the rest frames of particle-1 and particle-2, we define the corresponding frames $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, respectively.

For decay chain-2, we denote the rest frames of the relevant particles by:

● The decay $X \to 2+4$ is defined in the rest frame of particle-X, which is the same as in decay chain-1, and we denote it by $\boldsymbol{S}_X$.

● In the rest frame of particle-2, which is obtained by applying the helicity boost from the rest frame of particle-X, we define it as $\bar{\boldsymbol{S}}_2$.

● The second two-body decay $4 \to 1+Y$ is defined in the rest frame of particle-4, which we denote by $\bar{\boldsymbol{S}}_4$.

● By boosting from the rest frame of particle-4 to the rest frames of particle-1 and particle-Y, we define the corresponding frames $\bar{\boldsymbol{S}}_1$ and $\bar{\boldsymbol{S}}_Y$, respectively.

These different frames can be related via helicity boosts starting from $\boldsymbol{S}_X$. For decay chain 1, the relations among the frames are given by

$$\begin{aligned} &L_{i,h}^{-1}(\mathbf{q}_i^*)\boldsymbol{S}_X = \boldsymbol{S}_i \quad (i=3,Y), \\ &L_{j,h}^{-1}(\mathbf{q}_j^*)\boldsymbol{S}_3 = \boldsymbol{S}_j \quad (j=1,2). \end{aligned} \tag{481}$$

For decay chain-2, the relations among the frames are given by

$$\begin{aligned} &L_{i,h}^{-1}(\bar{\mathbf{q}}_i^*)\boldsymbol{S}_X = \bar{\boldsymbol{S}}_i \quad (i=4,2), \\ &L_{j,h}^{-1}(\bar{\mathbf{q}}_j^*)\bar{\boldsymbol{S}}_4 = \bar{\boldsymbol{S}}_j \quad (j=1,Y). \end{aligned} \tag{482}$$

Since the helicity of each final-state particle is defined along its momentum direction in the corresponding two-body decay COM frame, and since helicity is invariant under rotations and also invariant under boosts along that direction, the helicity of the particle remains unchanged after performing the helicity boost from the two-body decay COM frame to its rest frame. Equivalently, it becomes the z-axis direction in the particle's own rest frame. Therefore, starting from the final-state particle rest frames in the reference decay chain, one can use the sequence of frame transformations introduced above to boost step by step to the final-state particle rest frames in the other decay chain. The net effect of this sequence of helicity boosts is a pure rotation. This rotation is precisely the alignment rotation that must be applied to the non-reference decay chain.

For example, using Eqs. (481) and (482), for particle-1, the relation between its rest frames in the two decay chains, $\boldsymbol{S}_1$ and $\bar{\boldsymbol{S}}_1$, is given by

$$L_{1,h}^{-1}(\bar{\mathbf{q}}_1^*)L_{4,h}^{-1}(\bar{\mathbf{q}}_4^*)L_{3,h}(\mathbf{q}_3^*)L_{1,h}(\mathbf{q}_1^*)\boldsymbol{S}_1 = \bar{\boldsymbol{S}}_1, \tag{483}$$

From this, the relation between the two helicities λ_1 and $\bar{\lambda}_1$ can be determined by

$$L_{1,h}^{-1}(\bar{\mathbf{q}}_1^*)L_{4,h}^{-1}(\bar{\mathbf{q}}_4^*)L_{3,h}(\mathbf{q}_3^*)L_{1,h}(\mathbf{q}_1^*)|s_1\lambda_1\rangle = \sum_{\bar{\lambda}_1} D_{\lambda_1}^{\bar{\lambda}_1}(R_{\bar{1}1})|s_1\bar{\lambda}_1\rangle, \tag{484}$$

where $R_{\bar{1}1}$ denotes the alignment rotation that aligns $\bar{\lambda}_1$ of decay chain-2 with λ_1 of decay chain-1, and can be written as

$$R_{\bar{1}1} = L_{1,h}^{-1}(\bar{\mathbf{q}}_1^*)L_{4,h}^{-1}(\bar{\mathbf{q}}_4^*)L_{3,h}(\mathbf{q}_3^*)L_{1,h}(\mathbf{q}_1^*). \tag{485}$$

Similarly, the alignment rotations for particle-2 and particle-Y can be expressed as

$$R_{\bar{2}2} = L_{2,h}^{-1}(\bar{\mathbf{q}}_2^*)L_{3,h}(\mathbf{q}_3^*)L_{2,h}(\mathbf{q}_2^*), \tag{486}$$

$$R_{\bar{Y}Y} = L_{Y,h}^{-1}(\bar{\mathbf{q}}_Y^*)L_{4,h}^{-1}(\bar{\mathbf{q}}_4^*)L_{Y,h}(\mathbf{q}_Y^*). \tag{487}$$

After this alignment, the amplitude of decay chain-2 can be expressed as

$$\begin{aligned} &A_{\lambda_X}^{\lambda_1\lambda_2\lambda_Y}(\mathbf{k}_X,\bar{\mathbf{q}}_Y^*,\bar{\mathbf{q}}_1^*,\bar{\mathbf{q}}_2^*;4) \\ &= \sum_{\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_Y} A_{\lambda_X}^{\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_Y}(\mathbf{k}_X,\bar{\mathbf{q}}_Y^*,\bar{\mathbf{q}}_1^*,\bar{\mathbf{q}}_2^*;4)D_{\lambda_1}^{\bar{\lambda}_1*}(R_{\bar{1}1}) \\ &\times D_{\lambda_2}^{\bar{\lambda}_2*}(R_{\bar{2}2})D_{\lambda_Y}^{\bar{\lambda}_Y*}(R_{\bar{Y}Y}). \end{aligned} \tag{488}$$

Finally, the total amplitude is obtained by directly adding the amplitudes of the two decay chains and can be written as

$$\begin{aligned} A_{\lambda_X}^{\lambda_1\lambda_2\lambda_Y} &= A_{\lambda_X}^{\lambda_1\lambda_2\lambda_Y}(\mathbf{k}_X,\mathbf{q}_Y^*,\mathbf{q}_1^*,\mathbf{q}_2^*;3) \\ &\quad + A_{\lambda_X}^{\lambda_1\lambda_2\lambda_Y}(\mathbf{k}_X,\bar{\mathbf{q}}_Y^*,\bar{\mathbf{q}}_1^*,\bar{\mathbf{q}}_2^*;4) \\ &= A_{\lambda_X}^{\lambda_1\lambda_2\lambda_Y}(\mathbf{k}_X,\mathbf{q}_Y^*,\mathbf{q}_1^*,\mathbf{q}_2^*;3) \end{aligned}$$

$$+\sum_{\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_Y} A_{\lambda_X}^{\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_Y}(\mathbf{k}_X,\bar{\mathbf{q}}_Y^*,\bar{\mathbf{q}}_1^*,\bar{\mathbf{q}}_2^*;4)D_{\lambda_1}^{\bar{\lambda}_1*}(R_{\bar{1}1})$$
$$\times D_{\lambda_2}^{\bar{\lambda}_2*}(R_{\bar{2}2})D_{\lambda_Y}^{\bar{\lambda}_Y*}(R_{\bar{Y}Y}). \quad (489)$$

In a more general case with additional decay chains and more final particles, the procedure for handling the cascade decay amplitude involves choosing one decay chain as the reference chain and aligning the helicities of the final-state particles in all other decay chains with the helicities in the reference chain. Concretely, starting from the rest frames of the particles in a given decay chain, successive helicity boosts are applied step by step to reach the reference frame of the initial parent particle. From this frame, helicity boosts are then applied step by step to reach the rest frames of the particles in the reference chain. The detailed procedure can be written as

$$\bar{S}_i \xrightarrow{L_{i,h}^{-1}(\bar{\mathbf{q}}_i^*)} \cdots \xrightarrow{L_{j,h}^{-1}(\bar{\mathbf{q}}_j^*)} S_X \xrightarrow{L_{k,h}(\mathbf{q}_k^*)} \cdots \xrightarrow{L_{i,h}(\mathbf{q}_i^*)} S_i, \quad (490)$$

and the corresponding alignment rotation is given by:

$$R_{\bar{i}i} = L_{i,h}^{-1}(\bar{\mathbf{q}}_i^*)\cdots L_{j,h}^{-1}(\bar{\mathbf{q}}_j^*)L_{k,h}(\mathbf{q}_k^*)\cdots L_{i,h}(\mathbf{q}_i^*). \quad (491)$$

By applying the corresponding alignment rotations to the helicities of the final-state particles in the other decay chains, the helicities of all decay chains can be aligned. The total cascade decay amplitude is obtained by summing the amplitudes with aligned helicities.

2. Lab helicity convention for multiple decay chains

For a cascade process with multiple decay chains, each chain is defined in the same lab frame. Since the lab helicity convention of all final-state particles is defined with respect to the momentum direction of particles, the spin quantization axes are consistent among different chains. Therefore, one can first compute the cascade amplitude of each decay chain in the lab frame and then obtain the total amplitude by directly summing the chain amplitudes.

As an explicit example, we consider two decay chains:

- Decay chain-1: $X \to 3+Y$, followed by $3 \to 1+2$.
- Decay chain-2: $X \to 4+2$, followed by $4 \to 1+Y$.

In the lab frame, following the single chain construction discussed above and using Eq. (464), the corresponding cascade decay amplitudes are given by

$$A_{\lambda_X}^{\lambda_1\lambda_2\lambda_Y}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y;3)$$
$$=\sum_{\lambda_3} A_{\lambda_X}^{\lambda_3\lambda_Y}(\mathbf{p}_X,\mathbf{p}_3,\mathbf{p}_Y)P_3A_{\lambda_3}^{\lambda_1\lambda_2}(\mathbf{p}_3,\mathbf{p}_1,\mathbf{p}_2),$$
$$A_{\lambda_X}^{\lambda_1\lambda_2\lambda_Y}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y;4)$$
$$=\sum_{\lambda_4} A_{\lambda_X}^{\lambda_4\lambda_2}(\mathbf{p}_X,\mathbf{p}_4,\mathbf{p}_2)P_4A_{\lambda_4}^{\lambda_1\lambda_Y}(\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_Y). \quad (492)$$

Finally, the total cascade amplitude for this process is obtained by directly summing the two chain amplitudes as

$$A_{\lambda_X}^{\lambda_1\lambda_2\lambda_Y}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y) = A_{\lambda_X}^{\lambda_1\lambda_2\lambda_Y}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y;3)$$
$$+A_{\lambda_X}^{\lambda_1\lambda_2\lambda_Y}(\mathbf{p}_X,\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_Y;4). \quad (493)$$

Both conventions can be applied to cascade decays with multiple decay chains. In the two-body COM frame helicity convention, the step-by-step construction within each individual chain is consistent by construction: adjacent stages are connected by a single helicity boost, so the intermediate particle helicity is unchanged between the two corresponding rest frames, and no additional alignment is needed within the same chain. The mismatch only appears between different chains, and only on the external final-state helicity. Therefore, to coherently sum different chain amplitudes, it is sufficient to introduce, for each non-reference chain and for each final-state particle, a single alignment rotation $R_{\bar{i}i}$ and the associated Wigner D-matrices. Consequently, the number of alignment rotations to be determined scales roughly linearly with the number of non-reference chains times the number of final-state particles and is typically modest.

In contrast, in the lab frame helicity convention, all decay chains share the same helicity definition from the outset since each final-state helicity is always defined with respect to its lab momentum direction. Hence, different chains can be summed coherently without any extra chain-by-chain alignment rotations. The trade-off is that the building blocks written in the lab frame generally involve explicit Wigner rotations induced by non-collinear boosts, and such rotation factors may reappear at multiple steps in the cascade construction.

B. Covariant cascade decay

In this subsection, we primarily discuss the application of covariant approaches to cascade decays. Here, "covariant" means that the amplitude expression can be written directly in any frame, without additional input from the two-body COM frame. In our previous discussion, the covariant amplitude includes the **canonical-spinor amplitude** and the **covariant tensor amplitude (GS scheme)**.

Furthermore, in these constructions, the final-state particles are described using a common choice of spin quantization axis. Therefore, once a specific frame is selected in a cascade decay, for example the lab frame, the quantization axes for all two-body decays are taken to point in the same direction, such as the z-axis. As a result, these amplitudes allow for a unified treatment in cascade decays. In the following, we refer to this procedure as the covariant cascade decay.

In the covariant cascade decay, when the lab frame is chosen as the reference frame, the spin quantization axes of all two-body decay amplitudes are fixed to be the z-axis of this lab frame. Therefore, for a given decay chain, the two-body decay amplitude in which the intermediate particle appears as a daughter and the two-body amplitude in which it appears as a parent can be combined directly by summing over its spin index σ. Moreover, for different decay chains, one can directly add the chain amplitudes computed in the same lab frame to obtain the total cascade amplitude. This shows that, in this construction, no complicated alignment procedure is required, in contrast to the non-covariant treatment.

As a concrete example, consider the cascade decay process $X \to 3+Y$, followed by $3 \to 1+2$. In the lab frame, its cascade amplitude can be written as

$$\begin{aligned}&\mathcal{A}^{\sigma_1\sigma_2\sigma_Y}_{\sigma_X}(p_X,p_1,p_2,p_Y)\\ &=\sum_{\sigma_3}\mathcal{A}^{\sigma_3\sigma_Y}_{\sigma_X}(p_X,p_3,p_Y)P_3\mathcal{A}^{\sigma_1\sigma_2}_{\sigma_3}(p_3,p_1,p_2),\end{aligned} \tag{494}$$

where P_3 is the resonant propagator of the intermediate particle-3. Each two-body decay amplitude can be written as

$$\begin{aligned}\mathcal{A}^{\sigma_3\sigma_Y}_{\sigma_X}(p_X,p_3,p_Y)&=\sum_{L_1S_1}\mathcal{A}^{\sigma_3\sigma_Y}_{\sigma_X}(p_X,p_3,p_Y;L_1,S_1),\\ \mathcal{A}^{\sigma_1\sigma_2}_{\sigma_3}(p_3,p_1,p_2)&=\sum_{L_2S_2}\mathcal{A}^{\sigma_1\sigma_2}_{\sigma_3}(p_3,p_1,p_2;L_2,S_2),\end{aligned} \tag{495}$$

where (L_1,S_1) and (L_2,S_2) denote the possible LS partial wave combinations for the two two-body decays.

In the lab frame, the particle momenta are denoted by p_X, p_1, p_2, p_Y, and $p_3=p_1+p_2$. For each two-body decay, we directly substitute these momenta into the corresponding two-body amplitude. In this way, the cascade decay amplitude can be obtained directly. The lab frame momenta $\{p_X,p_1,p_2,p_3,p_Y\}$ shown in Fig. 12 can be used directly.

For processes with multiple decay chains, the amplitude for each chain can be obtained using the above procedure, without any additional boosts to intermediate COM frames and alignment rotations. As a result, the total amplitude is simply the coherent sum of the amplitudes from all decay chains. As an explicit example, we consider two decay chains,

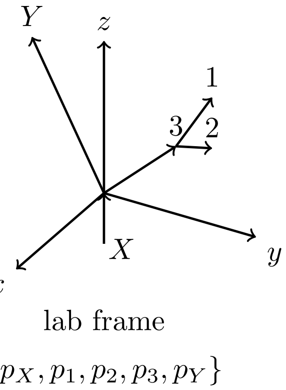

Fig. 12. Geometry of the cascade decay $X \to 3+Y$, $3 \to 1+2$ in the lab frame. The five lab frame four-momenta p_X,p_1,p_2,p_3,p_Y are used as the direct inputs for the two-body amplitudes.

- Decay chain-1: $X \to 3+Y$, followed by $3 \to 1+2$.
- Decay chain-2: $X \to 4+2$, followed by $4 \to 1+Y$.

The total amplitude can be written as

$$\begin{aligned}\mathcal{A}^{\sigma_1\sigma_2\sigma_Y}_{\sigma_X}(p_X,p_1,p_2,p_Y)=&\,\mathcal{A}^{\sigma_1\sigma_2\sigma_Y}_{\sigma_X}(p_X,p_1,p_2,p_Y;3)\\ &+\mathcal{A}^{\sigma_1\sigma_2\sigma_Y}_{\sigma_X}(p_X,p_1,p_2,p_Y;4),\end{aligned} \tag{496}$$

where the two terms denote the covariant cascade decay amplitudes associated with the two distinct resonances 3 and 4.

When more decay chains are present, one only needs to compute the amplitude for each chain in the lab frame and then add them directly to obtain the total amplitude. This shows that using covariant decay amplitudes as the basic building blocks for cascade decays leads to a more streamlined approach and is well-suited for handling cascade processes with multiple decay chains.

VI. THE CASCADE DECAY OF $\Lambda_c^+ \to \Lambda\pi^+\pi^0$

To validate and compare different PWA schemes, the cascade decay $\Lambda_c^+ \to \Lambda\pi^+\pi^0$ is used as a benchmark process, and fits are performed using different schemes. This cascade decay can proceed via several possible intermediate resonant states, with each resonance corresponding to a distinct decay chain.

In this work, the main objective is to test whether the coefficients in front of each partial wave, g_{LS}, and the resulting physical observables are consistent across different PWA schemes. For this reason, only the intermediate states listed in Ref. [85] with statistical significance larger than 5σ are included. The selected components are the $\rho(770)^+$, $\Sigma(1385)^+$, $\Sigma(1385)^0$, $\Sigma(1670)^+$, $\Sigma(1670)^0$,

$\Sigma(1750)^+$, and $\Sigma(1750)^0$ states, as well as the non-resonant (NR_{1^-}) component. These selected components can be organized into three distinct topological structures of decay chains, namely:

- **Topology I** (ρ^+-type): $\Lambda_c^+ \to \Lambda R$, $R \to \pi^+\pi^0$, $\Lambda \to p\pi^-$,
  where R is a meson resonance decaying into $\pi^+\pi^0$. The dominant contributions in this topology come from the $\rho(770)^+$ and the non-resonant $J^P = 1^-$ component, denoted as NR_{1^-}.

- **Topology II** (Σ^{*+}-type): $\Lambda_c^+ \to \Sigma^{*+}\pi^0$, $\Sigma^{*+} \to \Lambda\pi^+$, $\Lambda \to p\pi^-$,
  where Σ^{*+} is an excited baryon resonance. This topology includes the $\Sigma(1385)^+$, $\Sigma(1670)^+$, and $\Sigma(1750)^+$ states.

- **Topology III** (Σ^{*0}-type): $\Lambda_c^+ \to \Sigma^{*0}\pi^+$, $\Sigma^{*0} \to \Lambda\pi^0$, $\Lambda \to p\pi^-$,
  where Σ^{*0} is another excited baryon resonance. This topology includes the $\Sigma(1385)^0$, $\Sigma(1670)^0$, and $\Sigma(1750)^0$ states.

To analyze these decay channels, the open-source package TF-PWA [86] is used to generate events with the g_{LS} values provided in Ref. [85]. The generated events are then fitted using different partial wave amplitudes for comparison. Since only the consistency among the formalisms is of interest, the generated samples do not include background contributions or detector acceptance and efficiency effects.

A. Calculate amplitudes in different ways

In practice, we performed fits using the traditional-LS amplitude, the canonical-spinor amplitude, and the helicity amplitude, and found that they provide mutually consistent results. Therefore, for the components whose final-state quantum numbers are defined in the canonical basis, we use our canonical-spinor amplitude as an example and compare it with the helicity amplitude that is predominantly used in TF-PWA.

Next, we present several useful relations associated with the orbital angular momentum L. We consider the decay process $3 \to 1+2$, where the particle masses are m_3, m_1, and m_2. The magnitude of the momentum in the COM frame can be written as

$$q = \frac{\sqrt{((m_3+m_2)^2-m_1^2)((m_3+m_1)^2-m_2^2)}}{2m_3}. \tag{497}$$

The Blatt-Weisskopf barrier factor $B'_L(q,q_0,d)$ is given by [16]

$$\begin{aligned}
B'_0(q,q_0,d) &= 1,\\
B'_1(q,q_0,d) &= \sqrt{\frac{1+(q_0d)^2}{1+(qd)^2}},\\
B'_2(q,q_0,d) &= \sqrt{\frac{9+3(q_0d)^2+(q_0d)^4}{9+3(qd)^2+(qd)^4}},\\
B'_3(q,q_0,d) &= \sqrt{\frac{225+45(q_0d)^2+6(q_0d)^4+(q_0d)^6}{225+45(qd)^2+6(qd)^4+(qd)^6}},\\
B'_4(q,q_0,d) &= \sqrt{\frac{11025+1575(q_0d)^2+135(q_0d)^4+10(q_0d)^6+(q_0d)^8}{11025+1575(qd)^2+135(qd)^4+10(qd)^6+(qd)^8}}.
\end{aligned} \tag{498}$$

These functions effectively suppress the amplitude at high L, preventing the unphysical growth of high-L partial waves. The parameter d is a hadronic scale that characterizes the size of the interaction region. In this work, the radius parameter is fixed at $d = 0.73$ fm, following Ref. [87].

1. *Helicity scheme*

For the **helicity scheme**, the amplitude of each two-body decay in the COM frame is given by Eq. (75). In the cascade decay, each intermediate resonance is associated with a propagator, denoted generically by $P_\rho(m)$, $P_{\Sigma^{*+}}(m)$, $P_{\Sigma^{*0}}(m)$, $P_{NR_{1^-}}(m)$, and $P_\Lambda(m)$.

The propagator P includes different models. For the Λ and the non-resonant NR_{1^-} contribution, the propagators are taken to be unity. For the Σ^{*+} and Σ^{*0} resonances, the Breit-Wigner formalism is used:

$$P_{\Sigma^*}(m) = \frac{1}{m_0^2-m^2-im_0\Gamma(m)}, \tag{499}$$

where m is the invariant mass of the resonance and m_0 is the nominal resonance mass. The mass-dependent width is

$$\Gamma(m) = \Gamma_0 \left(\frac{q}{q_0}\right)^{2L+1} \frac{m_0}{m} B_L'^2(q, q_0, d). \tag{500}$$

For the $\rho(770)^+$ resonance, the Gounaris-Sakurai model [88] is used:

$$P_\rho(m) = \frac{1 + D\Gamma_0/m_0}{m_0^2 - m^2 + f(m) - i m_0 \Gamma(m)}, \tag{501}$$

where the $f(m)$ and D are defined as

$$f(m) = \Gamma_0 \frac{m_0^2}{q_0^3} \left[q^2 \left[h(m) - h(m_0) \right] + (m_0^2 - m^2) q_0^2 \frac{\mathrm{d}h}{\mathrm{d}m}|_{m_0} \right], \tag{502}$$

$$h(m) = \frac{2q}{\pi m} \ln\left(\frac{m + 2q}{2m_\pi}\right), \tag{503}$$

$$\frac{\mathrm{d}h}{\mathrm{d}m}|_{m_0} = h(m_0)\left[(8q_0^2)^{-1} - (2m_0^2)^{-1}\right] + (2\pi m_0^2)^{-1}, \tag{504}$$

$$D = \frac{f(0)}{\Gamma_0 m_0} = \frac{3}{\pi}\frac{m_\pi^2}{q_0^2} \ln\left(\frac{m_0 + 2q_0}{2m_\pi}\right) + \frac{m_0}{2\pi q_0} - \frac{m_\pi^2 m_0}{\pi q_0^3}. \tag{505}$$

Using the construction of helicity cascade decay amplitudes discussed in subsubsection V.A.2, the amplitude for each decay chain can be explicitly written as follows.

For topology I, the corresponding helicity amplitude can be expressed as

$$A^{\lambda_p}_{\lambda_{\Lambda_c^+}}(\rho) = \sum_{\lambda_\Lambda, \lambda_\rho} A^{\lambda_\Lambda \lambda_\rho}_{\lambda_{\Lambda_c^+}} P_\rho(M_{\pi^+\pi^0}) A_{\lambda_\rho} A^{\lambda_p}_{\lambda_\Lambda}, \tag{506}$$

$$A^{\lambda_p}_{\lambda_{\Lambda_c^+}}(NR_{1^-}) = \sum_{\lambda_\Lambda, \lambda_{NR_{1^-}}} A^{\lambda_\Lambda \lambda_{NR_{1^-}}}_{\lambda_{\Lambda_c^+}} A_{\lambda_{NR_{1^-}}} A^{\lambda_p}_{\lambda_\Lambda}. \tag{507}$$

For Topology II, the corresponding helicity amplitude can be expressed as

$$A^{\lambda_p^{(1)}}_{\lambda_{\Lambda_c^+}}(\Sigma^{*+}) = \sum_{\lambda_{\Sigma^{*+}}, \lambda_\Lambda} A^{\lambda_{\Sigma^{*+}}}_{\lambda_{\Lambda_c^+}} P_{\Sigma^{*+}}(M_{\Lambda\pi^+}) A^{\lambda_\Lambda}_{\lambda_{\Sigma^{*+}}} A^{\lambda_p^{(1)}}_{\lambda_\Lambda}. \tag{508}$$

For Topology III, the corresponding amplitude can be expressed as

$$A^{\lambda_p^{(2)}}_{\lambda_{\Lambda_c^+}}(\Sigma^{*0}) = \sum_{\lambda_{\Sigma^{*0}}, \lambda_\Lambda} A^{\lambda_{\Sigma^{*0}}}_{\lambda_{\Lambda_c^+}} P_{\Sigma^{*0}}(M_{\Lambda\pi^0}) A^{\lambda_\Lambda}_{\lambda_{\Sigma^{*0}}} A^{\lambda_p^{(2)}}_{\lambda_\Lambda}. \tag{509}$$

Based on the discussion in subsubsection 5.1.2, when constructing helicity amplitudes for processes with multiple decay chains, it is necessary to choose one decay chain as a reference and align all the other decay chains accordingly. In the present analysis, we take the topology I decay chain as the reference chain and align the remaining decay chains with respect to it. Since, among the final stable particles, only the proton carries nonzero spin, for the topology II decay chain, the helicity boost associated with the proton is given by

$$S_p^{(1)} \xrightarrow{L_h^{-1}(\mathbf{p}_p^{\Lambda(1)})} S_\Lambda^{(1)} \xrightarrow{L_h^{-1}(\mathbf{p}_{\Lambda(1)}^{\Sigma^{*+}})} S_{\Sigma^{*+}}^{(1)} \xrightarrow{L_h^{-1}(\mathbf{p}_{\Sigma^{*+}}^{\Lambda_c^+})} S_{\Lambda_c^+} \xrightarrow{L_h(\mathbf{p}_\Lambda^{\Lambda_c^+})} S_\Lambda \xrightarrow{L_h(\mathbf{p}_p^\Lambda)} S_p, \tag{510}$$

and the corresponding alignment rotation is

$$R_{\Sigma^{*+}\rho} = L_h^{-1}(\mathbf{p}_p^{\Lambda(1)}) L_h^{-1}(\mathbf{p}_{\Lambda(1)}^{\Sigma^{*+}}) L_h^{-1}(\mathbf{p}_{\Sigma^{*+}}^{\Lambda_c^+}) L_h(\mathbf{p}_\Lambda^{\Lambda_c^+}) L_h(\mathbf{p}_p^\Lambda). \tag{511}$$

For the topology III decay chain, the helicity boost of the proton is given by

$$S_p^{(2)} \xrightarrow{L_h^{-1}(\mathbf{p}_p^{\Lambda(2)})} S_\Lambda^{(2)} \xrightarrow{L_h^{-1}(\mathbf{p}_{\Lambda(2)}^{\Sigma^{*0}})} S_{\Sigma^{*0}}^{(2)} \xrightarrow{L_h^{-1}(\mathbf{p}_{\Sigma^{*0}}^{\Lambda_c^+})} S_{\Lambda_c^+} \xrightarrow{L_h(\mathbf{p}_\Lambda^{\Lambda_c^+})} S_\Lambda \xrightarrow{L_h(\mathbf{p}_p^\Lambda)} S_p, \tag{512}$$

and the corresponding alignment rotation is

$$R_{\Sigma^{*0}\rho} = L_h^{-1}(\mathbf{p}_p^{\Lambda(2)}) L_h^{-1}(\mathbf{p}_{\Lambda(2)}^{\Sigma^{*0}}) L_h^{-1}(\mathbf{p}_{\Sigma^{*0}}^{\Lambda_c^+}) L_h(\mathbf{p}_\Lambda^{\Lambda_c^+}) L_h(\mathbf{p}_p^\Lambda). \tag{513}$$

Therefore, after applying these alignment rotations to Eqs. (508) and (509), and then summing them with the contributions from Eqs. (506) and (507), the total helicity amplitude for the cascade decay can be written as

$$\begin{aligned} A^{\lambda_p}_{\lambda_{\Lambda_c^+}} &= \left(A^{\lambda_p}_{\lambda_{\Lambda_c^+}}(\rho) + A^{\lambda_p}_{\lambda_{\Lambda_c^+}}(NR_{1^-})\right) \\ &+ \sum_{\lambda_p^{(1)}} \left(\sum_{\Sigma^{*+}} A^{\lambda_p^{(1)}}_{\lambda_{\Lambda_c^+}}(\Sigma^{*+})\right) D^{(\frac{1}{2})*}_{\lambda_p^{(1)} \lambda_p}(R_{\Sigma^{*+}\rho}) \\ &+ \sum_{\lambda_p^{(2)}} \left(\sum_{\Sigma^{*0}} A^{\lambda_p^{(2)}}_{\lambda_{\Lambda_c^+}}(\Sigma^{*0})\right) D^{(\frac{1}{2})*}_{\lambda_p^{(2)} \lambda_p}(R_{\Sigma^{*0}\rho}). \end{aligned} \tag{514}$$

Then, this cascade decay amplitude can be fitted using the method described in subsection II.C to determine the corresponding LS partial wave coefficients g_{LS}.

2. *Canonical-spinor scheme*

The construction of the cascade decay amplitude in the **canonical-spinor scheme** is now discussed. The canonical-spinor two-body decay amplitudes used here are those of Eq. (344), with the normalization factor N set to 1. The corresponding spin and orbital parts, $\mathcal{S}$ and $\mathcal{Y}$, are given in Eqs. (345) and (347), respectively. The explicit analytic form of each two-body decay amplitude is provided in subsection 4.3 and can be identified according to the spin assignments of the particles involved.

In this scheme, the four-momenta of all particles in any frame are directly inserted into the corresponding canonical-spinor two-body decay amplitudes to obtain the contribution of each decay chain. The canonical-spinor cascade decay amplitudes for each decay chain can be written as

$$\mathcal{A}^{\sigma_p}_{\sigma_{\Lambda_c^+}}(\rho) = \sum_{\sigma_\Lambda,\sigma_\rho} \mathcal{A}^{\sigma_\Lambda\sigma_\rho}_{\sigma_{\Lambda_c^+}} P_\rho(M_{\pi^+\pi^0})\mathcal{A}_{\sigma_\rho}\mathcal{A}^{\sigma_p}_{\sigma_\Lambda}, \tag{515}$$

$$\mathcal{A}^{\sigma_p}_{\sigma_{\Lambda_c^+}}(NR_{1^-}) = \sum_{\sigma_\Lambda,\sigma_{NR_{1^-}}} \mathcal{A}^{\sigma_\Lambda\sigma_{NR_{1^-}}}_{\sigma_{\Lambda_c^+}} \mathcal{A}_{\sigma_{NR_{1^-}}}\mathcal{A}^{\sigma_p}_{\sigma_\Lambda}, \tag{516}$$

$$\mathcal{A}^{\sigma_p}_{\sigma_{\Lambda_c^+}}(\Sigma^{*+}) = \sum_{\sigma_{\Sigma^{*+}},\sigma_\Lambda} \mathcal{A}^{\sigma_{\Sigma^{*+}}}_{\sigma_{\Lambda_c^+}} P_{\Sigma^{*+}}(M_{\Lambda\pi^+})\mathcal{A}^{\sigma_\Lambda}_{\sigma_{\Sigma^{*+}}}\mathcal{A}^{\sigma_p}_{\sigma_\Lambda}, \tag{517}$$

$$\mathcal{A}^{\sigma_p}_{\sigma_{\Lambda_c^+}}(\Sigma^{*0}) = \sum_{\sigma_{\Sigma^{*0}},\sigma_\Lambda} \mathcal{A}^{\sigma_{\Sigma^{*0}}}_{\sigma_{\Lambda_c^+}} P_{\Sigma^{*0}}(M_{\Lambda\pi^0})\mathcal{A}^{\sigma_\Lambda}_{\sigma_{\Sigma^{*0}}}\mathcal{A}^{\sigma_p}_{\sigma_\Lambda}. \tag{518}$$

The propagators of the intermediate resonances have already been given in Eq. (499) and Eq. (501).

Following the discussion in subsection V.B, the total amplitude can be written as

$$\begin{aligned}\mathcal{A}^{\sigma_p}_{\sigma_{\Lambda_c^+}} &= \mathcal{A}^{\sigma_p}_{\sigma_{\Lambda_c^+}}(\rho) + \mathcal{A}^{\sigma_p}_{\sigma_{\Lambda_c^+}}(NR_{1^-}) \\ &\quad + \sum_{\Sigma^{*+}} \mathcal{A}^{\sigma_p}_{\sigma_{\Lambda_c^+}}(\Sigma^{*+}) + \sum_{\Sigma^{*0}} \mathcal{A}^{\sigma_p}_{\sigma_{\Lambda_c^+}}(\Sigma^{*0}).\end{aligned} \tag{519}$$

Using this cascade decay amplitude, a fit is performed following the procedure outlined in subsection VI.C, from which the corresponding LS partial wave coefficients and physical observables are extracted.

B. Numerical implementation of the canonical-spinor amplitude

In practical PWA, canonical-spinor amplitudes are calculated numerically. Some explicit examples are given in subsection IV.C; more generally, amplitudes for arbitrary spins are obtained by a direct numerical implementation of Eq. (332).

For the CGCs part, the three spins (s_1,s_2,s_3) determine the integers a,b,c appearing in Eq. (334). In the numerical implementation, the expression in Eq. (334) is expanded into a sum of products containing $(a+c)$ identity tensors δ^{a+c} and b antisymmetric tensors ε^b. The indices of each term are first reordered to match the ordering used in the expression, and each group of indices enclosed in $(\cdots)$ is then symmetrized. Multiplying by the overall prefactor in Eq. (334) finally yields the CGCs tensor with the desired set of $SU(2)$ indices.

For the orbital part $\mathcal{Y}$, the starting point is the basic $\langle 3_{I_1}|1|3_{I_2}]$ tensor. Its explicit form can be obtained by substituting the canonical-spinor variables and can be written as

$$\langle 3_{I_1}|1|3_{I_2}] = \lambda^\alpha_{I_1}(p_3)p_{1\alpha\dot\alpha}\tilde\lambda^{\dot\alpha}_{I_2}(p_3), \tag{520}$$

where the momenta are taken from Eq. (301), and the spinors are defined in Eq. (316) with $\tilde\lambda^{\dot\alpha}_I = \epsilon_{IJ}\tilde\lambda^{\dot\alpha J}$ between them. For higher L, a new tensor $\mathcal{Y}_{\{L_1\cdots L_{2L}\}}$ is constructed by taking the tensor product of L such $\langle 3_{I_1}|1|3_{I_2}]$ tensors, which yields an object with $2L$ indices. Symmetrizing over all $2L$ indices and multiplying by the overall prefactor in Eq. (347) then gives the orbital tensor.

For the spin part with total spin-S, a tensor τ is introduced. The basic tensor τ is defined in Eq. (346), and its explicit form in terms of canonical-spinor variables can be written as

$$\begin{aligned}(\tau'_i)^I_J(p_3,p_i) &= \frac{1}{\sqrt{2m_3(E_{i\text{com}}+m_i)}} \\ &\quad\times\left(\lambda^{I\alpha}(p_i)\lambda_{\alpha J}(p_3) - \tilde\lambda^I_{\dot\alpha}(p_i)\tilde\lambda^{\dot\alpha}_J(p_3)\right), \quad (i=1,2).\end{aligned} \tag{521}$$

The corresponding tensor τ^{2s_1} is constructed as the tensor product of $2s_1$ basic tensors τ_1, followed by complete symmetrization over its upper and lower indices. Particle 2 is treated analogously, yielding τ^{2s_2}. The tensors τ^{2s_1} and τ^{2s_2} are then contracted with the CGCs tensor over the appropriate indices to form the spin tensor $\mathcal{S}$.

Finally, $\mathcal{S}$ and $\mathcal{Y}$ are contracted with the external CGCs tensor to produce the amplitude with explicit $SU(2)$ indices. Using Eq. (344), this amplitude is then converted into the familiar form with σ indices.

C. Likelihood function and fit fraction

To perform the amplitude analysis, the TF-PWA package is used to generate phase-space (PHSP) samples for the multi-body final state, following the results given in Ref. [85]. These samples do not include background contributions or detector efficiency effects. The various amplitude formalisms discussed in this work are then fitted to these PHSP samples. The fitting procedure is as follows.

For each event, the probability density function is written as

$$P(x) = \frac{|A(x)|^2}{\int |A(x)|^2 \, \mathrm{d}\Phi}, \tag{522}$$

where x denotes a point in the final-state phase space. The squared amplitude is averaged over the spin states of the initial baryon Λ_c^+ as

$$|A(x)|^2 = \frac{1}{2} \sum_{\sigma_{\Lambda_c^+}, \sigma_p} \left| A^{\sigma_p}_{\sigma_{\Lambda_c^+}}(x) \right|^2, \tag{523}$$

with the factor $1/2$ accounting for the assumption of unpolarized Λ_c^+.

The integration is calculated using a Monte Carlo method with the generated PHSP samples:

$$\int |A(x)|^2 d\Phi \approx \frac{1}{N_{\rm MC}} \sum_{i=1}^{N_{\rm MC}} |A(x_i)|^2, \tag{524}$$

where x_i represents the i-th phase space point, and $N_{\rm MC}$ is the total number of MC samples.

The negative log-likelihood (NLL) is then constructed from the data as

$$-\ln L = -\sum_{i=1}^{N_{\rm data}} \ln P(x_i), \tag{525}$$

where the summation runs over all simulated data events. Minimizing this function yields the best-fit parameters of the model.

The parameter error matrix is calculated as the inverse of the Hessian matrix

$$V_{ij}^{-1} = -\frac{\partial^2 \ln L}{\partial X_i \partial X_j}, \tag{526}$$

where X_i is the i-th floating parameter in the fit.

The fit fraction (FF) for each resonant component can be calculated as

$$\mathrm{FF}_i = \frac{\int |A_i(x)|^2 \mathrm{d}\Phi}{\int \left| \sum_k A_k(x) \right|^2 \mathrm{d}\Phi}, \tag{527}$$

where $A_i(x)$ is the amplitude of the i-th component, and the integration is performed using the same PHSP samples.

In cases where interference between two components i and j is relevant, the interference fraction is defined as

$$\mathrm{FF}_{ij} = \frac{\int |A_i(x) + A_j(x)|^2 \mathrm{d}\Phi}{\int \left| \sum_k A_k(x) \right|^2 \mathrm{d}\Phi} - \mathrm{FF}_i - \mathrm{FF}_j. \tag{528}$$

The statistical uncertainties for the FFs are obtained using the standard form of error propagation:

$$\sigma_Y^2 = \sum_{i,j} \left(\frac{\partial Y}{\partial X_i} \right)_{X=\mu} \cdot V_{ij} \cdot \left(\frac{\partial Y}{\partial X_j} \right)_{X=\mu}, \tag{529}$$

where Y is the variable whose uncertainty is to be estimated, $X = \{X_i\}$ denotes the set of floating parameters, V_{ij} is the corresponding error matrix defined in Eq. (526), and μ is the result of the floating variables X.

D. Fitting result

To simplify the PWA fit, the global factor of the total amplitude for the full decay chain involving the $\rho(770)^+$ is fixed to have a magnitude of 1 and a phase of 0 in all schemes. Within each decay chain, one of the partial wave coefficients g_{LS} is likewise fixed to have a magnitude of 1 and a phase of 0. For the states Λ_c^+, $\rho(770)^+$, $\Sigma(1385)^+$, and $\Sigma(1385)^0$, the nominal mass and width are taken from the corresponding world average values [89]. In contrast, the resonance parameters of the other Σ^* states are adopted from the most recent measurements [90].

Following these conventions, each scheme involves a total of 38 floating parameters. These include 14 parameters associated with the magnitudes and phases of the global factor of the total amplitudes for each decay chain, and 24 parameters corresponding to partial wave coefficients.

The final fit results obtained in all schemes are mutually consistent. The fitted parameters and fit fractions are summarized in Table 1 and Table 2. It is worth noting that the FFs here are defined with normalization to the squared total amplitude; therefore, they are not simple probability fractions. Since the total amplitude contains interference terms among different components, destructive interference can reduce the denominator. As a result, the sum of the individual FFs can exceed 100%. The interference fractions and invariant mass spectra are shown in Table 3 and Fig. 13.

VII. CONCLUSION

In this paper, we study covariant amplitude constructions for PWA, aiming for decay amplitudes that are directly evaluable in any frame while maintaining a clear LS

Table 1. The numerical results of the total amplitude of different components and FFs. The total FF is 152.0%. Only statistical uncertainties are listed.

Process	Magnitude	Phase ϕ (rad)	FF (%)
$\Lambda\rho(770)^+$	1.0 (fixed)	0.0 (fixed)	58.3 ± 2.4
$\Sigma(1385)^+\pi^0$	0.49 ± 0.05	-0.11 ± 0.11	8.6 ± 0.5
$\Sigma(1385)^0\pi^+$	0.42 ± 0.05	3.06 ± 0.13	9.5 ± 0.5
$\Sigma(1670)^+\pi^0$	0.34 ± 0.06	-0.81 ± 0.16	4.0 ± 0.5
$\Sigma(1670)^0\pi^+$	0.35 ± 0.05	2.89 ± 0.17	2.6 ± 0.4
$\Sigma(1750)^+\pi^0$	1.89 ± 0.15	-1.71 ± 0.08	17.7 ± 1.8
$\Sigma(1750)^0\pi^+$	1.96 ± 0.17	1.39 ± 0.09	19.4 ± 1.9
$\Lambda+NR_{1^-}$	3.99 ± 0.29	2.17 ± 0.08	31.9 ± 2.9

partial wave expansion.

We first review the helicity and the canonical single-particle states and the corresponding two-particle states. In the two-body COM frame, when the two-particle state is coupled into a state with fixed total angular momentum J, the helicity and the canonical constructions correspond to two different coupling formalisms. In the helicity formalism, the coupling is performed directly in the helicity basis, whereas in the canonical formalism, the coupling is realized through the LS coupling. Consequently, for the two-body decay $3\to1+2$, there are two commonly used partial wave expansions of the decay amplitude, namely the helicity partial wave expansion and the LS partial wave expansion. Although these two expansions are written in different forms, they are physically equivalent. This equivalence follows from the completeness of the helicity two-particle states and the canonical two-particle states, which allows one basis to be expanded in the other.

We then derive the decay amplitudes in any frame for both the canonical formalism and the helicity formalism. Since the partial wave definitions are made in the COM frame, any frame evaluation requires boosting the momenta back to the COM frame. For non-collinear boosts, the boost induces non-trivial Wigner rotations, so that the amplitude in any frame is given by the COM frame amplitude accompanied by the Wigner rotations generated by the boost. In particular, in the LS partial wave framework that is central to this work, the traditional-LS coupling is defined in the COM frame. Therefore, the construction of the corresponding partial waves requires returning to the COM frame, and the definition is not mani-

Table 2. Numerical results of the partial wave coefficients g_{LS} for different resonances in the nominal fit are presented. Only statistical uncertainties are listed.

$\frac{1}{2}^+(\Lambda_c^+)\to\frac{3}{2}^+(\Sigma(1385)^+)+0^-(\pi^0)$			$\frac{1}{2}^+(\Lambda_c^+)\to\frac{3}{2}^+(\Sigma(1385)^0)+0^-(\pi^+)$		
Amplitude	Magnitude	Phase ϕ (rad)	Amplitude	Magnitude	Phase ϕ (rad)
$g^{\Sigma(1385)^+}_{1,\frac{3}{2}}$	1.0 (fixed)	0.0 (fixed)	$g^{\Sigma(1385)^0}_{1,\frac{3}{2}}$	1.0 (fixed)	0.0 (fixed)
$g^{\Sigma(1385)^+}_{2,\frac{3}{2}}$	1.22 ± 0.17	2.67 ± 0.12	$g^{\Sigma(1385)^0}_{2,\frac{3}{2}}$	1.61 ± 0.26	2.47 ± 0.13
$\frac{1}{2}^+(\Lambda_c^+)\to\frac{3}{2}^-(\Sigma(1670)^+)+0^-(\pi^0)$			$\frac{1}{2}^+(\Lambda_c^+)\to\frac{3}{2}^-(\Sigma(1670)^0)+0^-(\pi^+)$		
Amplitude	Magnitude	Phase ϕ (rad)	Amplitude	Magnitude	Phase ϕ (rad)
$g^{\Sigma(1670)^+}_{1,\frac{3}{2}}$	1.0 (fixed)	0.0 (fixed)	$g^{\Sigma(1670)^0}_{1,\frac{3}{2}}$	1.0 (fixed)	0.0 (fixed)
$g^{\Sigma(1670)^+}_{2,\frac{3}{2}}$	1.54 ± 0.29	0.99 ± 0.15	$g^{\Sigma(1670)^0}_{2,\frac{3}{2}}$	1.01 ± 0.20	0.25 ± 0.19
$\frac{1}{2}^+(\Lambda_c^+)\to\frac{1}{2}^-(\Sigma(1750)^+)+0^-(\pi^0)$			$\frac{1}{2}^+(\Lambda_c^+)\to\frac{1}{2}^-(\Sigma(1750)^0)+0^-(\pi^+)$		
Amplitude	Magnitude	Phase ϕ (rad)	Amplitude	Magnitude	Phase ϕ (rad)
$g^{\Sigma(1750)^+}_{0,\frac{1}{2}}$	1.0 (fixed)	0.0 (fixed)	$g^{\Sigma(1750)^0}_{0,\frac{1}{2}}$	1.0 (fixed)	0.0 (fixed)
$g^{\Sigma(1750)^+}_{1,\frac{1}{2}}$	0.35 ± 0.07	-2.26 ± 0.15	$g^{\Sigma(1750)^0}_{1,\frac{1}{2}}$	0.37 ± 0.07	-1.84 ± 0.14
$\frac{1}{2}^+(\Lambda_c^+)\to\frac{1}{2}^+(\Lambda)+1^-(\rho(770)^+)$			$\frac{1}{2}^+(\Lambda_c^+)\to\frac{1}{2}^+(\Lambda)+1^-(NR_{1^-})$		
Amplitude	Magnitude	Phase ϕ (rad)	Amplitude	Magnitude	Phase ϕ (rad)
$g^{\rho}_{0,\frac{1}{2}}$	1.0 (fixed)	0.0 (fixed)	$g^{NR}_{0,\frac{1}{2}}$	1.0 (fixed)	0.0 (fixed)
$g^{\rho}_{1,\frac{1}{2}}$	0.46 ± 0.08	-1.69 ± 0.09	$g^{NR}_{1,\frac{1}{2}}$	1.00 ± 0.08	-0.45 ± 0.09
$g^{\rho}_{1,\frac{3}{2}}$	0.89 ± 0.07	0.54 ± 0.08	$g^{NR}_{1,\frac{3}{2}}$	0.18 ± 0.06	-3.0 ± 0.4
$g^{\rho}_{2,\frac{3}{2}}$	0.55 ± 0.08	-0.14 ± 0.12	$g^{NR}_{2,\frac{3}{2}}$	0.29 ± 0.08	-2.22 ± 0.23
$\frac{1}{2}^+(\Lambda)\to\frac{1}{2}^+(p)+0^-(\pi^-)$					
Amplitude	Magnitude	Phase ϕ (rad)			
$g^{\Lambda}_{0,\frac{1}{2}}$	1.0 (fixed)	0.0 (fixed)			
$g^{\Lambda}_{1,\frac{1}{2}}$	0.435376 (fixed)	0.0 (fixed)			

Table 3. Interference fractions (I.F.) between Λ_c^+ amplitudes in units of percentage. The uncertainties are statistical only.

I.F.	$\Lambda+NR_{1^-}$	$\Sigma(1385)^0\pi^+$	$\Sigma(1385)^+\pi^0$	$\Sigma(1670)^0\pi^+$	$\Sigma(1670)^+\pi^0$	$\Sigma(1750)^0\pi^+$	$\Sigma(1750)^+\pi^0$
$\Sigma(1385)^0\pi^+$	-0.17 ± 0.28						
$\Sigma(1385)^+\pi^0$	-0.57 ± 0.28	-0.03 ± 0.02					
$\Sigma(1670)^0\pi^+$	-0.42 ± 0.12	-0.00 ± 0.00	-0.43 ± 0.05				
$\Sigma(1670)^+\pi^0$	-0.40 ± 0.12	-0.42 ± 0.06	0.00 ± 0.00	0.03 ± 0.01			
$\Sigma(1750)^0\pi^+$	-11.0 ± 2.2	0.03 ± 0.01	0.36 ± 0.04	-0.0 ± 0.0	-0.03 ± 0.03		
$\Sigma(1750)^+\pi^0$	-9.2 ± 2.1	0.26 ± 0.04	-0.06 ± 0.01	0.11 ± 0.04	0.07 ± 0.01	-7.1 ± 0.7	
$\Lambda\rho(770)^+$	-7.4 ± 2.5	-6.7 ± 0.4	-6.73 ± 0.33	0.56 ± 0.24	1.55 ± 0.25	-2.0 ± 0.9	-2.3 ± 0.9

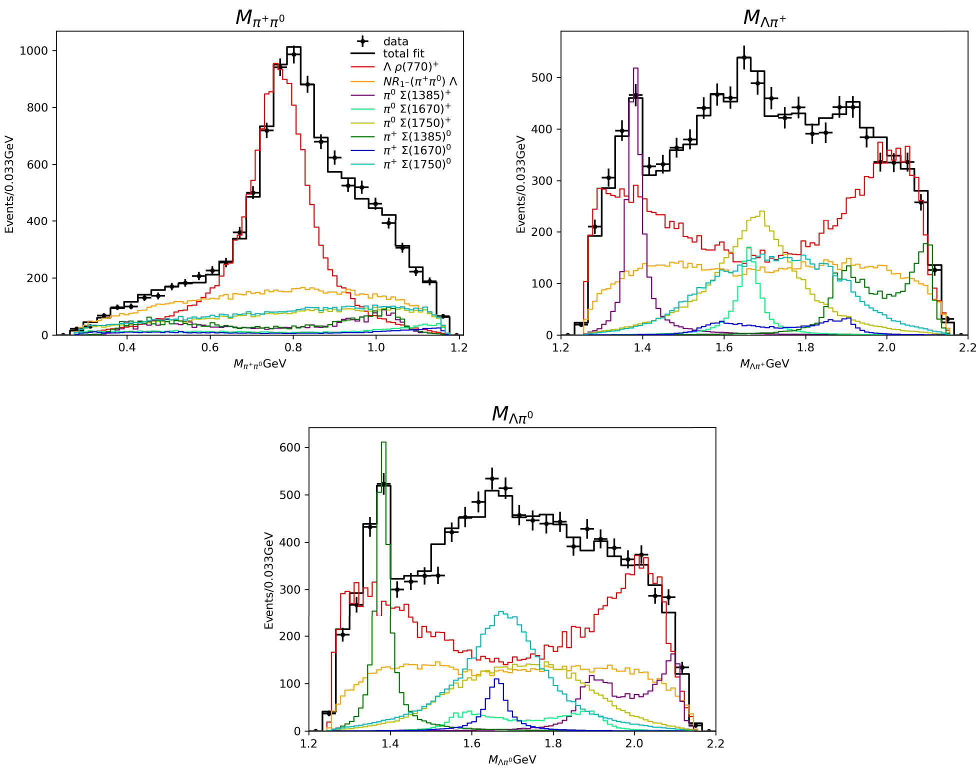

Fig. 13. (color online) Projections of the fit results in the invariant mass spectra $M_{\pi^+\pi^0}$, $M_{\Lambda\pi^+}$, and $M_{\Lambda\pi^0}$ are shown. Points with error bars denote the data, and different styles of the curves represent different components.

festly Lorentz covariant at the level of construction.

To restore Lorentz covariance, several methods, including Zemach tensor, covariant tensor, and covariant projection tensor methods, have been developed to construct Lorentz covariant LS tensors, in which the angular momentum coupling is implemented in Lorentz representations of $SO(3,1)$. In practice, tensor constructions of LS amplitudes can be organized into two classes, referred to as the PS scheme and the GS scheme.

• **PS scheme:** In the COM frame of the PS scheme, the partial wave amplitude can be decomposed into the covariant LS coupling structure and the pure-spin part.

$$A^{\sigma_1\sigma_2}_{\sigma_3}(k_3,p_1^*,p_2^*;L,S)=\underbrace{\Gamma^{\ell_1\ell_2}_{\ell_3}(k_3,p_1^*,p_2^*;L,S)}_{\text{covariant } LS \text{ coupling structure}}\times\underbrace{u^{\ell_3}_{\sigma_3}(k_3;s_3)\bar{u}^{\sigma_1}_{\ell_1}(k_1;s_1)\bar{u}^{\sigma_2}_{\ell_2}(k_2;s_2)}_{\text{pure-spin part}},\qquad(530)$$

where the pure-spin part does not depend on the final-state relative momentum. As a result, the spin coupling is equivalent to coupling the intrinsic spins defined at the rest frames of the individual particles, and it matches the traditional-LS method. For the orbital part, a Lorentz tensor structure in $SO(3,1)$ is constructed, which is equivalent, in the COM frame, to the spherical harmonic structure in the $SO(3)$ little group representation. Therefore, in the PS scheme, the orbital-spin separation is consistent with the traditional-LS method, realizing a clean separation between orbital and spin. The drawback is that the definition still relies on the COM frame, and hence it is not manifestly covariant at the level of construction.

• **GS scheme:** In the GS scheme, the partial wave amplitude can be decomposed into the covariant LS coupling structure and the general-spin part.

$$C^{\sigma_1\sigma_2}_{\sigma_3}(p_3,p_1,p_2;L,S) = \underbrace{\Gamma^{\ell_1\ell_2}_{\ell_3}(p_3,p_1,p_2;L,S)}_{\text{covariant } LS \text{ coupling structure}} \times \underbrace{u^{\ell_3}_{\sigma_3}(p_3;s_3)\bar{u}^{\sigma_1}_{\ell_1}(p_1;s_1)\bar{u}^{\sigma_2}_{\ell_2}(p_2;s_2)}_{\text{general-spin part}}, \tag{531}$$

which is manifestly covariant and can be evaluated directly in any frame, without boosting to the COM frame. However, when one returns to the COM frame, the general-spin part typically depends on the final-state momenta, so that orbital information enters the spin coupling, and the orbital-spin separation is incomplete. Consequently, this construction does not achieve the full orbital-spin separation of the traditional-LS method, and the physical meaning of the orbital-spin decomposition is modified. As a result, a fixed (L,S) component in the two definitions might not correspond to the same partial wave, which may lead to different partial wave coefficients. Only in the non-relativistic limit is the spin part reduced to rest frame spin wave functions, which are purely intrinsic-spin objects, and the tensor constructions of LS amplitudes in the GS scheme would have clear LS separation.

Therefore, current tensor constructions of LS amplitudes face a trade-off. It can be seen that neither scheme simultaneously provides a manifestly covariant construction that avoids boosts to the two-body COM frame and an LS separation identical to the traditional-LS definition. Additionally, for both schemes, when massless particles are involved, one must impose gauge invariance constraints explicitly and select a linearly independent set of vertex structures, which further increases the complexity of practical implementations.

Motivated by this trade-off, the goal is to construct an amplitude that satisfies the following requirements simultaneously. First, it is manifestly Lorentz covariant at the level of construction and can be evaluated directly in any frame. Second, it preserves, in physical meaning, the same orbital-spin separation as in the traditional-LS method, so that a fixed (L,S) component corresponds term-by-term to the traditional-LS decomposition. Third, the construction is given in a general formula applicable to particles of arbitrary spin, including the case with massless particles.

To achieve this, we present an on-shell construction of covariant orbital-spin coupling amplitudes referred to as the canonical-spinor amplitudes, which can be written as

$$\mathcal{A}^{\{I^{(1)}\},\{I^{(2)}\}}_{\{I^{(3)}\}}(p_3,p_1,p_2;L,S) = \underbrace{\mathcal{Y}^{L}_{\{I\}}}_{\text{orbital part}} \underbrace{\mathcal{S}^{\{I^{(1)}\},\{I^{(2)}\}}_{\{K\}} C^{S,\{K\};L,\{I\}}_{s_3,\{I^{(3)}\}}}_{\text{spin part}}. \tag{532}$$

In this scheme, the canonical-spinor variables are used as the basic building blocks to assemble the orbital part and the spin part. The angular momentum coupling is carried out in the little group $SU(2)$ representation, while the canonical-spinor variables carry SL(2,$\mathbb{C}$) Lorentz indices, which ensures the covariance of the construction. The resulting canonical-spinor amplitudes can be evaluated in any frame by directly substituting the four-momenta in that frame, without boosting each event back to the two-body COM frame. More importantly, when returning to the two-body COM frame, the LS definition in this construction coincides with that of the traditional-LS amplitude. This guarantees that the fitted g_{LS} coefficients agree with the traditional-LS method term by term and can therefore be compared directly.

In section 5, the treatment of cascade decays in non-covariant and covariant approaches is discussed. In non-covariant approaches, each two-body amplitude is defined in its own rest frame, so an explicit alignment of the spin quantization axis is required when assembling a multistep decay. In the canonical case, the spin quantization axis is fixed to the z-axis. The two-body amplitudes at different steps can therefore be aligned to a common reference z-axis, and different decay chains can be summed coherently. In the non-covariant helicity case, two standard treatments are used. One is to assemble the COM helicity amplitudes step by step and contract the intermediate-state indices. When multiple decay chains are present, the final-state helicity conventions can differ between chains, so an additional alignment is still needed before a coherent sum is formed. The other is to define all helicities with respect to a single reference frame, such as the lab, so that different chains can be added directly, in the same way as in the canonical treatment. In covariant approaches, the two-body building blocks are already expressed in a common covariant form and can be evaluated directly from the four-momenta. In practice, each chain amplitude is computed directly from the four-momenta, and the chain amplitudes are summed coherently without additional alignment steps. This leads to a simpler implementation, especially for processes with multiple decay chains.

As a benchmark, we study the cascade decay $\Lambda_c^+ \to \Lambda\pi^+\pi^0$ [85] and implement the amplitudes in TF-PWA.

We include the intermediate states with statistical significance larger than 5σ and perform fits using the likelihood function, fit fraction, and interference fraction definitions. The fitted magnitudes, phases, fit fractions, and interference fractions are consistent between the helicity amplitude, the traditional-*LS* amplitude, and the canonical-spinor amplitude. This numerical study supports that the canonical-spinor amplitude is practical for PWA.

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APPENDIX A: *D*-FUNCTIONS

This appendix summarizes the definition and basic properties of *D*-functions used throughout. The *D*-function with spin j is defined as

$$\begin{aligned} D^{\sigma'(j)}_{\sigma}(\alpha,\beta,\gamma) &= \langle \mathbf{k}, j\sigma'|U[R(\alpha,\beta,\gamma)]|\mathbf{k}, j\sigma\rangle \\ &= \mathrm{e}^{-\mathrm{i}\sigma'\alpha} d^{\sigma'(j)}_{\sigma}(\beta)\mathrm{e}^{-\mathrm{i}\sigma\gamma}. \end{aligned} \tag{A1}$$

The matrices $D^{\sigma'(j)}_{\sigma}$ are unitary and satisfy the group property:

$$\sum_{\sigma} D^{\sigma'(j)}_{\sigma}(R) D^{\sigma''(j)*}_{\sigma}(R) = \delta^{\sigma'}_{\sigma''}, \tag{A2}$$

$$D^{\sigma'(j)}_{\sigma''}(R_2R_1) = \sum_{\sigma} D^{\sigma'(j)}_{\sigma}(R_2) D^{\sigma(j)}_{\sigma''}(R_1). \tag{A3}$$

The functions $d^{\sigma'(j)}_{\sigma}(\beta)$ can be expressed as [91].

$$\begin{aligned} d^{\sigma'(j)}_{\sigma}(\beta) = \sum_t (-1)^t \frac{[(j+\sigma')!(j-\sigma')!(j+\sigma)!(j-\sigma)!]^{\frac{1}{2}}}{t!(j+\sigma'-t)!(j-\sigma-t)!(t+\sigma-\sigma')!} \\ \times \left(\cos\frac{\beta}{2}\right)^{2(j-t)+\sigma'-\sigma} \left(\sin\frac{\beta}{2}\right)^{2t+\sigma-\sigma'}. \end{aligned} \tag{A4}$$

The sum is taken over all values of t which lead to non-negative factorials.

The functions $d^{\sigma'(j)}_{\sigma}(\beta)$ have the following symmetry properties:

$$d^{\sigma'(j)}_{\sigma}(\beta) = (-1)^{\sigma'-\sigma} d^{\sigma(j)}_{\sigma'}(\beta) = (-1)^{\sigma'-\sigma} d^{\sigma'(j)}_{\sigma}(-\beta), \tag{A5}$$

$$d^{\sigma'(j)}_{\sigma}(\beta) = (-1)^{\sigma'-\sigma} d^{-\sigma'(j)}_{-\sigma}(\beta), \tag{A6}$$

$$d^{\sigma'(j)}_{\sigma}(\pi-\beta) = (-1)^{j+\sigma'} d^{\sigma'(j)}_{-\sigma}(\beta), \tag{A7}$$

$$d^{\sigma'(j)}_{\sigma}(\pi) = (-1)^{j-\sigma} \delta^{\sigma'}_{-\sigma}. \tag{A8}$$

Using Eq. (A7), the *D*-functions have

$$D^{\sigma'(j)*}_{\sigma}(R) = D^{\sigma(j)}_{\sigma'}(R^{-1}) = (-1)^{\sigma'-\sigma} D^{-\sigma'(j)}_{-\sigma}(R). \tag{A9}$$

Using the identity

$$R(\pi+\alpha,\pi-\beta,\pi-\gamma) = R(\alpha,\beta,\gamma)R(0,\pi,0), \tag{A10}$$

one obtains the corresponding relation for *D*-functions:

$$D^{\sigma'(j)}_{\sigma}(\pi+\alpha,\pi-\beta,\pi-\gamma) = (-1)^{j-\sigma} D^{\sigma'(j)}_{-\sigma}(\alpha,\beta,\gamma), \tag{A11}$$

and

$$D^{\sigma'(j)}_{\sigma}(\pi+\phi,\pi-\theta,0) = (-1)^{j} D^{\sigma'(j)}_{-\sigma}(\phi,\theta,0). \tag{A12}$$

The spherical harmonics $Y^L_{\sigma_L}(\phi,\theta)$ are related to the *D*-functions via

$$D^{\sigma_L(j)*}_{0}(\phi,\theta,0) = \sqrt{\frac{4\pi}{2L+1}} Y^L_{\sigma_L}(\theta,\phi). \tag{A13}$$

The *D*-functions satisfy the following coupling rule:

$$D^{\sigma'_1(j_1)}_{\sigma_1} D^{\sigma'_2(j_2)}_{\sigma_2} = \sum_{j_3} C^{j_1,\sigma'_1;j_2,\sigma'_2}_{j_3,\sigma'_3} C^{j_3,\sigma_3}_{j_1,\sigma_1;j_2\sigma_2} D^{\sigma'_3(j_3)}_{\sigma_3}, \tag{A14}$$

where

$$|j_1-j_2| \le j_3 \le j_1+j_2, \quad \sigma_3 = \sigma_1+\sigma_2, \quad \sigma'_3 = \sigma'_1+\sigma'_2. \tag{A15}$$

Or, equivalently,

$$D^{\sigma'_1(j_1)}_{\sigma_1} D^{\sigma'_3(j_3)*}_{\sigma_3} = \sum_{j_2} \frac{2j_2+1}{2j_3+1} C^{j_1,\sigma'_1;j_2,\sigma'_2}_{j_3,\sigma'_3} C^{j_3,\sigma_3}_{j_1,\sigma_1;j_2\sigma_2} D^{\sigma'_2(j_2)*}_{\sigma_2}, \tag{A16}$$

where

The action of a rotation on a particle state is given by:

$$|j_1-j_3| \le j_2 \le j_1+j_3, \quad \sigma_2 = \sigma_3-\sigma_1, \quad \sigma'_2 = \sigma'_3-\sigma'_1. \tag{A17}$$

$$U(R)|j\sigma\rangle = \sum_{\sigma'} D^{\sigma'(j)}_{\sigma}(R)|j\sigma'\rangle, \tag{A18}$$

or

$$\langle j\sigma|U^{\dagger}(R) = \sum_{\sigma'} D^{\sigma'(j)*}_{\sigma}(R)\langle j\sigma'| = \sum_{\sigma'} D^{\sigma(j)}_{\sigma'}(R^{-1})\langle j\sigma'|. \tag{A19}$$

APPENDIX B: NORMALIZATION OF ONE- AND TWO-PARTICLE STATES

One-particle momentum eigenstates $|\mathbf{p}, j\sigma\rangle$ and $|\mathbf{p}, j\lambda\rangle$ are normalized as

$$\begin{aligned}\langle \mathbf{p}', j'\sigma'|\mathbf{p}, j\sigma\rangle &= \tilde{\delta}(\mathbf{p}'-\mathbf{p})\,\delta_{jj'}\delta_{\sigma\sigma'},\\ \langle \mathbf{p}', j'\lambda'|\mathbf{p}, j\lambda\rangle &= \tilde{\delta}(\mathbf{p}'-\mathbf{p})\,\delta_{jj'}\delta_{\lambda\lambda'},\end{aligned} \tag{B1}$$

where the Lorentz invariant δ-function is given by

$$\tilde{\delta}(\mathbf{p}'-\mathbf{p}) = (2\pi)^3(2E)\,\delta^{(3)}(\mathbf{p}'-\mathbf{p}), \tag{B2}$$

and the invariant volume element is defined by

$$\tilde{\mathrm{d}}p = \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3\,2E}. \tag{B3}$$

With these conventions, the completeness relations take the form:

$$\sum_{j,\sigma}\int \tilde{d}p\,|\mathbf{p}, j\sigma\rangle\langle\mathbf{p}, j\sigma| = \mathbf{I}, \qquad \sum_{j,\lambda}\int \tilde{d}p\,|\mathbf{p}, j\lambda\rangle\langle\mathbf{p}, j\lambda| = \mathbf{I}, \tag{B4}$$

where $\mathbf{I}$ denotes the identity operator.

For the discussion of two-particle kinematics in this appendix, only spinless particles are considered. The spin label is therefore suppressed, and the notation $|\mathbf{p}\rangle$ is used for one-particle states. Two-particle states are defined by

$$|\mathbf{p}_1\mathbf{p}_2\rangle \equiv |\mathbf{p}_1\rangle|\mathbf{p}_2\rangle, \tag{B5}$$

and satisfy the normalization

$$\langle \mathbf{p}_1'\mathbf{p}_2'|\mathbf{p}_1\mathbf{p}_2\rangle = \tilde{\delta}(\mathbf{p}_1'-\mathbf{p}_1)\tilde{\delta}(\mathbf{p}_2'-\mathbf{p}_2). \tag{B6}$$

A two-particle system with four-momenta p_1 and p_2 can be equivalently described by the total four-momentum

$$p = p_1 + p_2, \tag{B7}$$

and by the orientation Ω of the relative three-momentum in the COM frame. The invariant mass of the two-particle system is denoted by

$$m = E_{1\mathrm{com}} + E_{2\mathrm{com}}, \tag{B8}$$

and the magnitude of the three-momentum in the COM frame is denoted by q, such that

$$\mathbf{p}_1^* = -\mathbf{p}_2^*, \qquad q = |\mathbf{p}_1^*| = |\mathbf{p}_2^*|. \tag{B9}$$

Two-particle states labeled by the total four-momentum p and the solid angle Ω of the relative momentum are introduced as

$$|p,\Omega\rangle = N|\mathbf{p}_1\mathbf{p}_2\rangle, \tag{B10}$$

where N is the normalization constant. The desired normalization of these states is

$$\langle p',\Omega'|p,\Omega\rangle = (2\pi)^4\delta^{(4)}(p'-p)\,\delta^{(2)}(\Omega'-\Omega), \tag{B11}$$

where $\delta^{(2)}(\Omega'-\Omega)$ denotes the delta function on the unit sphere, normalized according to

$$\int \mathrm{d}\Omega\delta^{(2)}(\Omega'-\Omega) = 1. \tag{B12}$$

The relation between $|\mathbf{p}_1\mathbf{p}_2\rangle$ and $|p,\Omega\rangle$ is determined by the two-body phase-space measure. In the COM frame, the two-body phase space is defined by

$$\begin{aligned}\mathrm{d}\Phi &= (2\pi)^4\delta^{(4)}(p_1+p_2-p)\tilde{\mathrm{d}}p_1\,\tilde{\mathrm{d}}p_2\\ &= \frac{1}{(4\pi)^2}\frac{q}{m}\,\mathrm{d}\Omega.\end{aligned} \tag{B13}$$

Combining this with $\delta^{(2)}(\Omega'-\Omega)$ and integrating over these variables gives:

$$\begin{aligned}\int \delta^{(2)}(\Omega'-\Omega)\,\mathrm{d}\Phi &= \int \delta^{(2)}(\Omega'-\Omega)\,\frac{1}{(4\pi)^2}\frac{q}{m}\,\mathrm{d}\Omega\\ &= \frac{1}{(4\pi)^2}\frac{q}{m}.\end{aligned} \tag{B14}$$

Multiplying Eqs. (B6) and (B11) by the invariant volume element $\mathrm{d}\tilde{p}_1\,\mathrm{d}\tilde{p}_2$, Eq. (B6) integrates to 1. For Eq. (B11), using Eqs. (B10), (B13) and (B14), one obtains the normalization constant.

$$N = \frac{1}{4\pi}\sqrt{\frac{q}{m}}. \tag{B15}$$

APPENDIX C: CLEBSCH-GORDAN COEFFICIENTS

C.1. CGCs calculation

Consider two canonical states $|s_1\sigma_1\rangle$ and $|s_2\sigma_2\rangle$. Our goal is to couple these two states into a total spin-S, such that

$$|s_1 - s_2| \leq S \leq s_1 + s_2. \tag{C1}$$

The states are related by a linear relation given by

$$|S\sigma\rangle = \sum_{\sigma_1\sigma_2} C_{S,\sigma}^{s_1,\sigma_1;s_2,\sigma_2} |s_1\sigma_1\rangle|s_2\sigma_2\rangle, \tag{C2}$$

where the coefficients $C_{S,\sigma}^{s_1,\sigma_1;s_2,\sigma_2}$ are called CGCs. We have $\sigma = \sigma_1 + \sigma_2$. An expression for the CG coefficients was also derived by Wigner [91]

$$\begin{aligned}
&C_{S,\sigma}^{s_1,\sigma_1;s_2,\sigma_2} = \delta_{\sigma}^{\sigma_1+\sigma_2} \\
&\times \left[(2S+1)\frac{(S+s_1-s_2)!(S-s_1+s_2)!(s_1+s_2-S)!(S+\sigma)!(S-\sigma)!}{(S+s_1+s_2+1)!(s_1-\sigma_1)!(s_1+\sigma_1)!(s_2-\sigma_2)!(s_2+\sigma_2)!}\right]^{\frac{1}{2}} \\
&\times \sum_t \frac{(-1)^{t+S+\sigma}}{t!} \frac{(S+\sigma_1+s_2-t)!(s_1-\sigma_1+t)!}{S-(s_1+s_2-t)!(S+\sigma-t)!(t+s_1-s_2-\sigma)!}.
\end{aligned} \tag{C3}$$

The sum is taken over all values of t that lead to non-negative factorials. Using the orthogonality relations, we can derive the inverse relations as

$$|s_1\sigma_1\rangle|s_2\sigma_2\rangle = \sum_S C_{s_1,\sigma_1;s_2,\sigma_2}^{S,\sigma} |S\sigma\rangle. \tag{C4}$$

The CGCs have the following symmetry properties:

$$C_{S,\sigma}^{s_1,\sigma_1;s_2,\sigma_2} = (-1)^{s_1+s_2-S} C_{S,-\sigma}^{s_1,-\sigma_1;s_2,-\sigma_2}, \tag{C5}$$

$$C_{S,\sigma}^{s_1,\sigma_1;s_2,\sigma_2} = (-1)^{s_1+s_2-S} C_{S,\sigma}^{s_2,\sigma_2;s_1,\sigma_1}, \tag{C6}$$

$$C_{S,\sigma}^{s_1,\sigma_1;s_2,\sigma_2} = (-1)^{s_1-\sigma_1} \left[\frac{2S+1}{2s_2+1}\right]^{\frac{1}{2}} C_{s_2,-\sigma_2}^{s_1,\sigma_1;S,-\sigma}. \tag{C7}$$

C.2. Examples of $SU(2)$ CGCs calculation

We first present some examples of $SU(2)$ CGC calculations. The $SU(2)$ CGCs are

$$\begin{aligned}
C_{s,\{J\}}^{s_1,\{I\};s_2,\{K\}} \equiv &\sqrt{\frac{(a+b)!(b+c)!(a+c+1)!}{b!(a+b+c+1)!a!c!}} \\
&\times \delta_{(J_1\cdots J_a}^{(I_1\cdots I_a} \varepsilon^{I_{a+1}\cdots I_{a+b}),(K_1\cdots K_b} \delta_{J_{a+1}\cdots J_{a+c})}^{K_{b+1}\cdots K_{b+c})},
\end{aligned} \tag{C8}$$

with

$$\begin{aligned}
a &= s_1 + s - s_2, \\
b &= s_1 + s_2 - s, \\
c &= s_2 + s - s_1, \\
\varepsilon^{I_1\cdots I_b,K_1\cdots K_b} &= \varepsilon^{I_1K_1}\cdots\varepsilon^{I_bK_b}.
\end{aligned} \tag{C9}$$

1. $s = \frac{1}{2}$, $s_1 = \frac{1}{2}$, $s_2 = 0$ with $a = 1$, $b = 0$, $c = 0$. The CGC is

$$C_{\frac{1}{2},\{J_1\}}^{\frac{1}{2},\{I_1\};0} = \delta_{J_1}^{I_1}. \tag{C10}$$

2. $s = 1$, $s_1 = 1$, $s_2 = 0$ with $a = 2$, $b = 0$, $c = 0$. The CGC is

$$C_{1,\{J_1J_2\}}^{1,\{I_1I_2\};0} = \delta_{(J_1J_2)}^{(I_1I_2)} = \frac{1}{2}(\delta_{J_1}^{I_1}\delta_{J_2}^{I_2} + \delta_{J_1}^{I_2}\delta_{J_2}^{I_1}). \tag{C11}$$

3. $s = \frac{3}{2}, s_1 = \frac{3}{2}, s_2 = 0$ with $a = 3, b = 0, c = 0$. The CGC is

$$\begin{aligned}
C_{\frac{3}{2},\{J_1J_2J_3\}}^{\frac{3}{2},\{I_1I_2I_3\};0} = \delta_{(J_1J_2J_3)}^{(I_1I_2I_3)} = &\frac{1}{6}(\delta_{J_1}^{I_1}\delta_{J_2}^{I_2}\delta_{J_3}^{I_3} + \delta_{J_1}^{I_1}\delta_{J_2}^{I_3}\delta_{J_3}^{I_2} \\
&+ \delta_{J_1}^{I_2}\delta_{J_2}^{I_1}\delta_{J_3}^{I_3} + \delta_{J_1}^{I_2}\delta_{J_2}^{I_3}\delta_{J_3}^{I_1} + \delta_{J_1}^{I_3}\delta_{J_2}^{I_1}\delta_{J_3}^{I_2} + \delta_{J_1}^{I_3}\delta_{J_2}^{I_2}\delta_{J_3}^{I_1}).
\end{aligned} \tag{C12}$$

4. $s = \frac{1}{2}, s_1 = \frac{1}{2}, s_2 = 1$ with $a = 0, b = 1, c = 1$. The CGC is

$$\begin{aligned}
C_{\frac{1}{2},\{J_1\}}^{\frac{1}{2},\{I_1\};1,\{K_1K_2\}} &= \sqrt{\frac{2}{3}}\varepsilon^{I_1(K_1}\delta_{J_1}^{K_2)} \\
&= \frac{1}{2}\sqrt{\frac{2}{3}}\left(\varepsilon^{I_1K_1}\delta_{J_1}^{K_2} + \varepsilon^{I_1K_2}\delta_{J_1}^{K_1}\right).
\end{aligned} \tag{C13}$$

5. for $s = \frac{3}{2}, s_1 = \frac{1}{2}, s_2 = 1$ with $a = 1, b = 0, c = 2$. The CGC is

$$\begin{aligned}
C_{\frac{3}{2},\{J_1J_2J_3\}}^{\frac{1}{2},\{I_1\};1,\{K_1K_2\}} = \delta_{(J_1J_2J_3)}^{I_1(K_1K_2)} = &\frac{1}{6}(\delta_{J_1}^{I_1}\delta_{J_2}^{K_1}\delta_{J_3}^{K_2} + \delta_{J_1}^{I_1}\delta_{J_3}^{K_1}\delta_{J_2}^{K_2} \\
&+ \delta_{J_2}^{I_1}\delta_{J_1}^{K_1}\delta_{J_3}^{K_2} + \delta_{J_2}^{I_1}\delta_{J_3}^{K_1}\delta_{J_1}^{K_2} + \delta_{J_3}^{I_1}\delta_{J_1}^{K_1}\delta_{J_2}^{K_2} \\
&+ \delta_{J_3}^{I_1}\delta_{J_2}^{K_1}\delta_{J_1}^{K_2}).
\end{aligned} \tag{C14}$$

6. The given values are $s = \frac{1}{2}, s_1 = \frac{3}{2}, s_2 = 1$ with

$a=1, b=2, c=0$. The CGC is

$$C^{\frac{3}{2},\{I_1I_2I_3\};1,\{K_1K_2\}}_{\frac{1}{2},\{J_1\}} = \sqrt{\frac{1}{2}}\delta^{(I_1}_{J_1}\varepsilon^{I_2I_3),(K_1K_2)}$$
$$= \frac{1}{6}\sqrt{\frac{1}{2}}(\delta^{I_1}_{J_1}\varepsilon^{I_2K_1}\varepsilon^{I_3K_2}+\delta^{I_1}_{J_1}\varepsilon^{I_3K_1}\varepsilon^{I_2K_2}$$
$$+\delta^{I_2}_{J_1}\varepsilon^{I_1K_1}\varepsilon^{I_3K_2}+\delta^{I_2}_{J_1}\varepsilon^{I_3K_1}\varepsilon^{I_1K_2}$$
$$+\delta^{I_3}_{J_1}\varepsilon^{I_1K_1}\varepsilon^{I_2K_2}+\delta^{I_3}_{J_1}\varepsilon^{I_2K_1}\varepsilon^{I_1K_2}. \quad \text{(C15)}$$

7. $s=1$, $s_1=1$, $s_2=1$, with $a=1$, $b=1$, and $c=1$. The CGC is

$$C^{1,\{I_1I_2\};1,\{K_1K_2\}}_{1,\{J_1J_2\}} = \delta^{(I_1}_{(J_1}\varepsilon^{I_2)(K_1}\delta^{K_2)}_{J_2)}$$
$$= \frac{1}{8}(\delta^{I_1}_{J_1}\varepsilon^{I_2K_1}\delta^{K_2}_{J_2}+\delta^{I_1}_{J_1}\varepsilon^{I_2K_2}\delta^{K_1}_{J_2}+\delta^{I_2}_{J_1}\varepsilon^{I_1K_1}\delta^{K_2}_{J_2}$$
$$+\delta^{I_2}_{J_1}\varepsilon^{I_1K_2}\delta^{K_1}_{J_2}+\delta^{I_1}_{J_2}\varepsilon^{I_2K_1}\delta^{K_2}_{J_1}+\delta^{I_1}_{J_2}\varepsilon^{I_2K_2}\delta^{K_1}_{J_1}$$
$$+\delta^{I_2}_{J_2}\varepsilon^{I_1K_1}\delta^{K_2}_{J_1}+\delta^{I_2}_{J_2}\varepsilon^{I_1K_2}\delta^{K_1}_{J_1}). \quad \text{(C16)}$$

APPENDIX D: NUMERICAL BASES-TRANSFORMATION

In this appendix, numerical procedures for relating different partial wave bases are summarized. Although amplitudes may take different explicit forms depending on the chosen basis, the underlying physical content remains the same. In particular, the same scattering or decay process can be described in different schemes, such as the helicity basis or the traditional-LS basis.

Consider two coupling bases with partial wave amplitudes denoted by C_{LS} and $H_{L'S'}$, respectively. They are connected by a linear transformation with basis-change coefficients $t_{LS;L'S'}$,

$$C_{LS} = \sum_{L'S'} t_{LS;L'S'} H_{L'S'}, \quad \text{(D1)}$$

where $t_{LS;L'S'}$ is the transformation matrix element from the $H_{L'S'}$ basis to the C_{LS} basis. In practical calculations, one possible method to determine the transformation matrix $t_{LS;L'S'}$ is through sampling in phase space. If there are N independent LS combinations, one can select N sets of kinematically allowed momenta from the physical phase space, denoted as:

$$\{p_3^1p_1^1p_2^1\}, \quad \{p_3^2p_1^2p_2^2\}, \quad \cdots, \quad \{p_3^Np_1^Np_2^N\}. \quad \text{(D2)}$$

By calculating each amplitude at these sampled momentum points, one can construct and solve a system of linear equations to extract the numerical values of $t_{LS;L'S'}$. Since the physical amplitude A is basis-independent, it can be equivalently written in either representation:

$$A = \sum_{LS} g_{LS} C_{LS} = \sum_{L'S'} g_{L'S'} H_{L'S'}, \quad \text{(D3)}$$

where $g_{L,S}$ and $g_{L'S'}$ are the energy-dependent coefficients in their respective bases. Assuming that g_{LS} are known in the C_{LS} basis, one can straightforwardly compute the transformed coefficients $g_{L'S'}$ in the $H_{L'S'}$ basis via:

$$g'_{L'S'} = \sum_{LS} g_{LS} t_{LS;L'S'}. \quad \text{(D4)}$$

This transformation ensures that the physical amplitude remains invariant under the change of basis, thereby guaranteeing the consistency of physical predictions such as decay widths, angular distributions, and differential cross sections.

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