Chinese Physics C   Vol. 50, No. 9 (2026)

Frame Dependence in Generalized Chiral Kinetic Theory[*]

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Abstract: We investigate the frame dependence of distribution functions within the framework of generalized chiral kinetic theory. Based on the derived transformation rules governing the choice of frame, we analytically obtain the global equilibrium solutions in the presence of vorticity and electromagnetic fields. Our results show that, under the assumption of varying electromagnetic fields, these equilibrium solutions can be uniquely determined.

Keywords: relativistic heavy-ion collisions, quark gluon plasma, chiral magnetic effect, chiral vortical effect

DOI: 10.1088/1674-1137/ae740d   **CSTR:**

I. INTRODUCTION

Recently, novel chiral and spin effects have attracted significant attention in relativistic heavy-ion collisions, such as the chiral magnetic effect [1–3], chiral vortical effect [4–7], global polarization [8–14], spin alignment [15–17], and so on. These various chiral and spin effects have greatly stimulated theoretical research, especially in the context of quantum kinetic theory. In recent years, quantum kinetic theory has been developed in various directions, including extensions from massless [18–32] to massive fermions [33–41], from Abelian to non-Abelian frameworks [26, 42–45], from first- to second-order formulations [29, 46–51], from flat to curved spacetime [48, 52–54], from collisionless to collisional dynamics [24, 55–65], and from fermions to gauge bosons [54, 66–70]. Recent reviews on quantum kinetic theory can be found in Refs. [71–76].

In quantum kinetic theory, distribution functions can exhibit nontrivial dependence on the frame in which they are defined. For instance, in chiral kinetic theory (CKT) for massless fermions, the chiral distribution functions depend on the choice of frame [23, 24]. When transforming the distribution function from one frame to another, a nontrivial side-jump term must be included to preserve Lorentz invariance. In our previous work [41], a generalized chiral kinetic theory (GCKT) for fermions of arbitrary mass was derived. This framework provides a convenient formalism for describing the quantum transport of arbitrary-mass fermions and ensures a smooth transition between massive and massless cases. However, in [41], the transformation rules for distribution functions across different frames were not examined. Moreover, the global-equilibrium solutions were not uniquely determined, leading to differing results in [41] and [33]. In the present work, we devote special attention to the frame-dependence transformation rules for distribution functions within the GCKT. Based on the derived transformation rules, we analytically obtain the global-equilibrium solution in the presence of vorticity and electromagnetic (EM) fields. Our results demonstrate that, under the assumption of varying EM fields, these equilibrium solutions can be uniquely determined.

In Section II, we provide a brief review of the Wigner function formalism for massive Dirac fermions. Section III presents the main results of the GCKT. In Section IV, we derive the transformation rules governing the frame dependence of the distribution functions. Section V is devoted to determining the Wigner functions in global equilibrium under the influence of vorticity and EM fields. In Section VI, we demonstrate that these equilibrium Wigner functions can be uniquely determined under varying EM fields. Finally, a summary of our findings is given in Section VII.

Throughout this work, we employ the Minkowski metric convention $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and Levi-Civita tensor convention $\epsilon^{0123} = 1$. We adopt natural units with $\hbar = c = 1$ unless otherwise stated.

Received 21 January 2026; Accepted 28 May 2026

[*] This work was supported in part by the National Natural Science Foundation of China under Grant Nos. 12575147, 12321005, and 12175123

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II. WIGNER FUNCTIONS AND EQUATIONS

In the Wigner function framework [77–80], the Wigner function $W(x,p)$ for Dirac fermions in a background EM field is defined as the ensemble average of the gauge-invariant Wigner operator:

$$W(x,p)=\int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\langle\bar{\psi}(x_+)U(x_+,x_-)\psi(x_-)\rangle,$$

where $x_\pm\equiv x\pm y/2$ and U denotes the Wilson line.

$$U(x_+,x_-)\equiv e^{-iy^\mu\int_0^1 dsA_\mu\left(x-\frac{1}{2}y+sy\right)}$$

which ensures gauge invariance. The electric charge has been absorbed into the gauge potential A_μ. The Wigner function is a matrix in spinor space and can be decomposed as

$$W=\frac{1}{4}(\mathscr{F}+i\gamma^5\mathscr{P}+\gamma^\mu\mathscr{V}_\mu+\gamma^5\gamma^\mu\mathscr{A}_\mu+\frac{1}{2}\sigma^{\mu\nu}\mathscr{S}_{\mu\nu}).$$

In chiral kinetic theory (CKT), or its generalized version GCKT, chiral Wigner functions are introduced as

$$\mathscr{J}_s^\mu=\frac{1}{2}(\mathscr{V}^\mu+s\mathscr{A}^\mu),$$

where $s=\pm$ denotes right-/left-handed chirality. With these functions, the Wigner equations can be organized into two groups. Group 1 reads

$$\Pi_\mu\mathscr{J}_s^\mu=\frac{1}{2}m\mathscr{F},\tag{1}$$

$$-\hbar\nabla_\mu\mathscr{J}_s^\mu=ms\mathscr{P},\tag{2}$$

$$\hbar\left(\nabla^\mu\mathscr{J}_s^\nu-\nabla^\nu\mathscr{J}_s^\mu\right)-2s\epsilon^{\mu\nu\rho\sigma}\Pi_\rho\mathscr{J}_{s\sigma}=m\mathscr{S}^{\mu\nu},\tag{3}$$

and Group 2 reads

$$\Pi^\mu\mathscr{F}+\frac{1}{2}\hbar\nabla_\nu\mathscr{S}^{\mu\nu}=m\sum_s\mathscr{J}_s^\mu,\tag{4}$$

$$\Pi^\mu\mathscr{P}+\frac{1}{2}\hbar\nabla_\nu\tilde{\mathscr{S}}^{\mu\nu}=0,\tag{5}$$

$$\hbar\nabla_\mu\mathscr{P}-2\Pi^\nu\tilde{\mathscr{S}}_{\mu\nu}=2m\sum_s s\mathscr{J}_s^\mu,\tag{6}$$

$$\hbar\nabla_\mu\mathscr{F}-2\Pi^\nu\mathscr{S}_{\mu\nu}=0,\tag{7}$$

where $\tilde{\mathscr{S}}^{\mu\nu}=\epsilon^{\mu\nu\alpha\beta}\mathscr{S}_{\alpha\beta}/2$. In the background-field approximation, the operators ∇^μ and Π^μ are given by

$$\begin{aligned}\nabla^\mu\equiv&\partial_x^\mu-j_0\left(\frac{1}{2}\hbar\partial^p\cdot\partial_x\right)F^{\mu\nu}\partial_\nu^p,\\ \Pi^\mu\equiv&p^\mu-\frac{1}{2}\hbar j_1\left(\frac{1}{2}\hbar\partial^p\cdot\partial_x\right)F^{\mu\nu}\partial_\nu^p,\end{aligned}$$

where $j_0(z)$ and $j_1(z)$ are spherical Bessel functions, and ∂_x acts only on the field strength tensor, not on the Wigner function. To prepare for the semiclassical expansion in the next section, we have restored the explicit dependence on $\hbar$ in the Wigner equations and operator definitions. In the chiral limit, Group 1 decouples from Group 2 and gives rise to the chiral kinetic equation [29].

III. GENERALIZED CHIRAL KINETIC THEORY

This section reviews the derivation of the GCKT to first order in $\hbar$ and presents the derivation in greater detail and in a more stepwise manner than Ref.[41]. The Wigner functions can be expanded in powers of $\hbar$ as

$$\mathscr{J}_s^\mu=\mathscr{J}_s^{(0)\mu}+\hbar\mathscr{J}_s^{(1)\mu}+\hbar^2\mathscr{J}_s^{(2)\mu}+\cdots,$$

with analogous expansions for the other Wigner functions, $\mathscr{F}$, $\mathscr{P}$, and $\mathscr{S}_{\mu\nu}$. To first order, the operators ∇^μ and Π^μ reduce to

$$\nabla^\mu=\partial_x^\mu-F^{\mu\nu}\partial_\nu^p,\quad\Pi^\mu=p^\mu.$$

To disentangle the Wigner equations, we introduce a timelike 4-vector n^μ normalized as $n^2=1$. For simplicity, n^μ is taken to be constant and independent of x and p. In Section VI, we will show that the GCKT formulated with a general n^μ can be recovered through the frame-dependent transformation of the distribution functions. Using n^μ, any 4-vector X^μ can be decomposed as $X^\mu=X_nn^\mu+\bar{X}^\mu$, where $X_n=X\cdot n$ and $\bar{X}^\mu=\Delta^{\mu\nu}X_\nu$ with $\Delta^{\mu\nu}=g^{\mu\nu}-n^\mu n^\nu$. Similarly, the antisymmetric tensors $F^{\mu\nu}$ and $\mathscr{S}^{\mu\nu}$ can be expressed in terms of the electric and magnetic fields, and in terms of the electric-moment distribution function $\mathscr{K}^\mu$ and the magnetic-moment distribution function $\mathscr{M}^\mu$, respectively:

$$\begin{aligned}F^{\mu\nu}=&E^\mu n^\nu-E^\nu n^\mu-\bar{\epsilon}^{\mu\nu\sigma}B_\sigma,\\ \mathscr{S}^{\mu\nu}=&\mathscr{K}^\mu n^\nu-\mathscr{K}^\nu n^\mu-\bar{\epsilon}^{\mu\nu\sigma}\mathscr{M}_\sigma,\end{aligned}\tag{8}$$

where we have defined the purely spatial antisymmetric tensor $\bar{\epsilon}_{\mu\alpha\beta}=\epsilon_{\mu\nu\alpha\beta}n^\nu$. The inverse relations are given by

$$
\begin{aligned}
E^{\mu} =& F^{\mu\nu} n_{\nu}, \ \ B^{\mu} = \tilde{F}^{\mu\nu} n_{\nu}, \\
\mathscr{K}^{\mu} =& \mathscr{S}^{\mu\nu} n_{\nu}, \ \ \mathscr{M}^{\mu} = \tilde{\mathscr{S}}^{\mu\nu} n_{\nu},
\end{aligned}
$$

where $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}/2$. By substituting these expansions into the Wigner equation and extracting the terms order by order, we can derive the Wigner equations at each order. Note that, to obtain the GCKT to first order, we require the Wigner equations up to second order.

A. Wigner equations at zeroth order

At zeroth order, using the decomposition in terms of n^{μ}, Group 1 can be rewritten as

$$p_n \mathscr{J}_{sn}^{(0)} + \bar{p}_{\mu} \bar{\mathscr{J}}_{s}^{(0)\mu} = \frac{1}{2} m \mathscr{F}^{(0)}, \tag{9}$$

$$0 = ms\mathscr{P}^{(0)}, \tag{10}$$

$$2s\left(\bar{p}^{\mu} \mathscr{J}_{ns}^{(0)} - p_n \bar{\mathscr{J}}_{s}^{(0)\mu}\right) = m\mathscr{M}^{(0)\mu}, \tag{11}$$

$$2s\left(\bar{p}^{\mu} \bar{\mathscr{J}}_{s}^{(0)\nu} - \bar{p}^{\nu} \bar{\mathscr{J}}_{s}^{(0)\mu}\right) = m\bar{\epsilon}^{\mu\nu\beta} \mathscr{K}_{\beta}^{(0)}, \tag{12}$$

where the last two equations correspond to the timelike-spacelike and spacelike-spacelike components of the tensor equation (3) along the direction n^{μ}, respectively. In Group 2, all equations are vector equations and can be decomposed into timelike and spacelike components as

$$p_n \mathscr{F}^{(0)} = m \sum_{s} \mathscr{J}_{sn}^{(0)}, \tag{13}$$

$$\bar{p}^{\mu} \mathscr{F}^{(0)} = m \sum_{s} \bar{\mathscr{J}}_{s}^{(0)\mu}, \tag{14}$$

$$p_n \mathscr{P}^{(0)} = 0, \tag{15}$$

$$\bar{p}^{\mu} \mathscr{P}^{(0)} = 0, \tag{16}$$

$$\bar{p}^{\nu} \mathscr{M}_{\nu}^{(0)} = m \sum_{s} s \mathscr{J}_{sn}^{(0)}, \tag{17}$$

$$-p_n \mathscr{M}_{\mu}^{(0)} - \bar{p}^{\nu} \bar{\epsilon}_{\mu\nu\beta} \mathscr{K}^{(0)\beta} = m \sum_{s} s \bar{\mathscr{J}}_{s\mu}^{(0)}, \tag{18}$$

$$\bar{p}^{\nu} \mathscr{K}_{\nu}^{(0)} = 0, \tag{19}$$

$$-p_n \mathscr{K}_{\mu}^{(0)} + \bar{p}^{\nu} \bar{\epsilon}_{\mu\nu\beta} \mathscr{M}^{(0)\beta} = 0. \tag{20}$$

From Eqs. (11), (13), (15), and (20), we obtain, respectively,

$$\bar{\mathscr{J}}_{s}^{(0)\mu} = \frac{1}{p_n} \bar{p}^{\mu} \mathscr{J}_{sn}^{(0)} - \frac{sm}{2p_n} \mathscr{M}^{(0)\mu}, \tag{21}$$

$$\mathscr{F}^{(0)} = \frac{m}{p_n} \sum_{s} \mathscr{J}_{sn}^{(0)}, \tag{22}$$

$$\mathscr{P}^{(0)} = 0, \tag{23}$$

$$\mathscr{K}_{\mu}^{(0)} = \frac{1}{p_n} \bar{\epsilon}_{\mu\alpha\beta} p^{\alpha} \mathscr{M}^{(0)\beta}. \tag{24}$$

It is straightforward to verify that, with these equations, Eqs. (10), (12), (14), (16), and (19) are all automatically satisfied. Substituting Eqs. (21) and (22) into Eq. (9), together with Eq. (17), we obtain the on-shell condition for $\mathscr{J}_{sn}^{(0)}$

$$\frac{p^2 - m^2}{p_n} \mathscr{J}_{sn}^{(0)} = 0.$$

Thus, the general solution can be written as

$$\mathscr{J}_{sn}^{(0)} = p_n \mathcal{J}_{sn}^{(0)} \delta\left(p^2 - m^2\right), \tag{25}$$

where $\mathcal{J}_{sn}^{(0)}$ is an arbitrary function regular at $p^2 = m^2$. Substituting Eqs. (21) and (24) into Eq. (18), together with Eq. (17), we obtain the on-shell condition for $\mathscr{M}_{\mu}^{(0)}$

$$\frac{p^2 - m^2}{p_n} \mathscr{M}_{\mu}^{(0)} = 0.$$

Thus, the general solution can be written as

$$\mathscr{M}_{\mu}^{(0)} = p_n \mathcal{M}_{\mu}^{(0)} \delta\left(p^2 - m^2\right), \tag{26}$$

where $\mathcal{M}_{\mu}^{(0)}$ is an arbitrary function regular at $p^2 = m^2$. Using the relation (8), the antisymmetric tensor $\mathscr{S}^{(0)\mu\nu}$ and its dual tensor $\tilde{\mathscr{S}}^{(0)\mu\nu}$ can be expressed, respectively, as

$$
\begin{aligned}
\mathscr{S}^{(0)\mu\nu} =& \frac{1}{p_n} \epsilon^{\mu\nu\alpha\beta} p_{\alpha} \mathscr{M}_{\beta}, \\
\tilde{\mathscr{S}}^{(0)\mu\nu} =& \frac{1}{p_n} \left(\mathscr{M}^{(0)\mu} p^{\nu} - \mathscr{M}^{(0)\nu} p^{\mu}\right).
\end{aligned}
\tag{27}
$$

All the equations above indicate that we can choose $\mathscr{J}_{sn}^{(0)}$ and $\mathscr{M}_{\mu}^{(0)}$ as the basic distribution functions, since all other Wigner functions can be expressed in terms of these functions.

The only remaining equation, Eq.(17), implies that the longitudinal component of the spacelike vector $\mathscr{M}^{(0)}_{\mu}$ in the direction of $\bar{p}_\mu$ is not independent. Hence, we can decompose the magnetic moments $\mathscr{M}^{\mu}$ into parts parallel to and orthogonal to the spacelike momentum $\bar{p}^{\mu}$.

$$\mathscr{M}^{(0)\mu} = \mathscr{M}^{(0)\mu}_{\parallel} + \mathscr{M}^{(0)\mu}_{\perp}, \tag{28}$$

$$\mathscr{M}^{(0)\mu}_{\parallel} = p_n \mathcal{M}^{(0)\mu}_{\parallel} \delta(p^2 - m^2), \tag{29}$$

$$\mathscr{M}^{(0)\mu}_{\perp} = p_n \mathcal{M}^{(0)\mu}_{\perp} \delta(p^2 - m^2), \tag{30}$$

with the relations

$$\mathcal{M}^{(0)\mu}_{\parallel} = \frac{m}{\bar{p}^2} \bar{p}^{\mu} \sum_s s \mathcal{J}^{(0)}_{sn}, \tag{31}$$

$$\bar{p}_\mu \mathcal{M}^{(0)\mu}_{\perp} \delta(p^2 - m^2) = 0. \tag{32}$$

Therefore, we can identify $\mathcal{J}^{(0)}_{sn}$ and $\mathcal{M}^{(0)\mu}_{\perp}$ as the final set of basic distribution functions. Whether we use $\mathcal{J}^{(0)}_{sn}$ and $\mathcal{M}^{(0)\mu}_{\perp}$ or $\mathcal{J}^{(0)}_{sn}$ and $\mathcal{M}^{(0)\mu}$ as the basic variables is a matter of convenience, depending on the specific form of the expressions.

B. Wigner equations at first order

At first order, using the decomposition with respect to n^{μ}, Group 1 can be rewritten as

$$p_n \mathscr{J}^{(1)}_{sn} + \bar{p}_\mu \bar{\mathscr{J}}^{(1)\mu}_{s} = \frac{1}{2} m \mathscr{F}^{(1)}, \tag{33}$$

$$-\nabla_\mu \mathscr{J}^{(0)\mu}_{s} = ms \mathscr{P}^{(1)}, \tag{34}$$

$$2s(\bar{p}^{\mu} \mathscr{J}^{(1)}_{ns} - p_n \bar{\mathscr{J}}^{(1)\mu}_{s}) + \bar{\epsilon}^{\mu\rho\sigma} \nabla_\rho \mathscr{J}^{(0)}_{s\sigma} = m \mathscr{M}^{(1)\mu}, \tag{35}$$

$$\begin{aligned} &2s(\bar{p}^{\mu} \bar{\mathscr{J}}^{(1)\nu}_{s} - \bar{p}^{\nu} \bar{\mathscr{J}}^{(1)\mu}_{s}) + \bar{\epsilon}^{\mu\nu\rho} n^{\sigma} (\nabla_\rho \mathscr{J}^{(0)}_{s\sigma} - \nabla_\sigma \mathscr{J}^{(0)}_{s\rho}) \\ &= m \bar{\epsilon}^{\mu\nu\beta} \mathscr{K}^{(1)}_{\beta}, \end{aligned} \tag{36}$$

and Group 2 can be decomposed into timelike and spacelike components as

$$p_n \mathscr{F}^{(1)} + \frac{1}{2} n_\mu \nabla_\nu \mathscr{S}^{(0)\mu\nu} = m \sum_s \mathscr{J}^{(1)}_{sn}, \tag{37}$$

$$\bar{p}^{\mu} \mathscr{F}^{(1)} + \frac{1}{2} \Delta^{\mu}{}_{\lambda} \nabla_\nu \mathscr{S}^{(0)\lambda\nu} = m \sum_s \bar{\mathscr{J}}^{(1)\mu}_{s}, \tag{38}$$

$$p_n \mathscr{P}^{(1)} + \frac{1}{2} n_\mu \nabla_\nu \tilde{\mathscr{S}}^{(0)\mu\nu} = 0, \tag{39}$$

$$\bar{p}^{\mu} \mathscr{P}^{(1)} + \frac{1}{2} \Delta^{\mu}{}_{\lambda} \nabla_\nu \tilde{\mathscr{S}}^{(0)\lambda\nu} = 0, \tag{40}$$

$$n^{\mu} \nabla_\mu \mathscr{P}^{(0)} + 2\bar{p}^{\nu} \mathscr{M}^{(1)}_{\nu} = 2m \sum_s s \mathscr{J}^{(1)}_{sn}, \tag{41}$$

$$\bar{\nabla}_\mu \mathscr{P}^{(0)} - 2p_n \mathscr{M}^{(1)}_{\mu} - 2\bar{\epsilon}_{\mu\nu\beta} \bar{p}^{\nu} \mathscr{K}^{(1)\beta} = 2m \sum_s s \bar{\mathscr{J}}^{(1)}_{s\mu}, \tag{42}$$

$$n^{\mu} \nabla_\mu \mathscr{F}^{(0)} + 2\bar{p}^{\nu} \mathscr{K}^{(1)}_{\nu} = 0, \tag{43}$$

$$\bar{\nabla}_\mu \mathscr{F}^{(0)} - 2p_n \mathscr{K}^{(1)}_{\mu} + 2\bar{\epsilon}_{\mu\nu\beta} \bar{p}^{\nu} \mathscr{M}^{(1)\beta} = 0. \tag{44}$$

From Eqs. (35), (37), (39), and (44), we obtain, respectively,

$$\bar{\mathscr{J}}^{(1)\mu}_{s} = \frac{\bar{p}^{\mu}}{p_n} \mathscr{J}^{(1)}_{sn} - \frac{sm}{2p_n} \mathscr{M}^{(1)\mu} + \frac{s}{2p_n} \bar{\epsilon}^{\mu\rho\sigma} \nabla_\rho \mathscr{J}^{(0)}_{s\sigma}, \tag{45}$$

$$\mathscr{F}^{(1)} = \frac{m}{p_n} \sum_s \mathscr{J}^{(1)}_{sn} - \frac{1}{2p_n} n_\mu \nabla_\nu \mathscr{S}^{(0)\mu\nu}, \tag{46}$$

$$\mathscr{P}^{(1)} = -\frac{1}{2p_n} n_\mu \nabla_\nu \tilde{\mathscr{S}}^{(0)\mu\nu}, \tag{47}$$

$$\mathscr{K}^{(1)}_{\mu} = \frac{1}{p_n} \bar{\epsilon}_{\mu\alpha\beta} p^{\alpha} \mathscr{M}^{(1)\beta} + \frac{1}{2p_n} \Delta_{\mu}{}^{\lambda} \nabla_\lambda \mathscr{F}^{(0)}. \tag{48}$$

From Eq.(41), we obtain

$$\bar{p}^{\nu} \mathscr{M}^{(1)}_{\nu} = m \sum_s s \mathscr{J}^{(1)}_{sn}, \tag{49}$$

where we have used Eq. (23). Substituting Eq. (47) into Eq. (34) and using Eqs. (21) and (27) yields

$$p^{\mu} \nabla_\mu \left(\frac{\mathscr{J}^{(0)}_{sn}}{p_n} \right) = \frac{ms}{2p_n^2} E_\mu \mathscr{M}^{(0)\mu}, \tag{50}$$

which is the generalized chiral kinetic equation (GCKE) for $\mathscr{J}^{(0)}_{sn}$. Similarly, by substituting Eq. (47) into Eq. (40) and using Eq. (24) or Eq. (27), one obtains the GCKE for $\mathscr{M}^{(0)\mu}$.

$$p^\nu\nabla_\nu\left(\frac{\mathscr{M}^{(0)\mu}}{p_n}\right)=\left(\frac{\bar{p}^\mu}{p_n}E^\nu-\bar{\epsilon}^{\mu\nu\alpha}B_\alpha\right)\frac{\mathscr{M}^{(0)}_\nu}{p_n}. \tag{51}$$

Inserting Eqs. (45) and (46) into Eq. (33) and using Eqs. (49), (21), and (27), we obtain

$$\frac{p^2-m^2}{p_n}\mathscr{J}^{(1)}_{sn}=\frac{s}{2p_n}\bar{\epsilon}^{\mu\rho\sigma}F_{\rho\sigma}\bar{\mathscr{J}}^{(0)}_{s\mu}. \tag{52}$$

From Eqs. (25) and (26), we obtain the following general expression for $\mathscr{J}^{(1)}_{sn}$:

$$\begin{aligned}\mathscr{J}^{(1)}_{sn}&=p_n\mathcal{J}^{(1)}_{sn}\delta(p^2-m^2)\\&\quad-\frac{s}{2}\bar{\epsilon}^{\mu\rho\sigma}F_{\rho\sigma}\left(\bar{p}_\mu\mathcal{J}^{(0)}_{sn}-\frac{sm}{2}\mathcal{M}^{(0)}_\mu\right)\delta'(p^2-m^2).\end{aligned} \tag{53}$$

Substituting Eqs. (45) and (48) into Eq. (42) and using Eqs. (49), (21), and (22), we obtain

$$\frac{p^2-m^2}{p_n}\mathscr{M}^{(1)}_\mu=\frac{m}{2p_n}\bar{\epsilon}_{\mu\rho\sigma}F^{\rho\sigma}\sum_s\left(\frac{1}{p_n}\mathscr{J}^{(0)}_{sn}\right). \tag{54}$$

From Eq. (25), we obtain the general expression

$$\begin{aligned}\mathscr{M}^{(1)}_\mu&=p_n\mathcal{M}^{(1)}_\mu\delta\left(p^2-m^2\right)\\&\quad-\frac{m}{2}\bar{\epsilon}_{\mu\rho\sigma}F^{\rho\sigma}\sum_s\mathcal{J}^{(0)}_{sn}\delta'(p^2-m^2).\end{aligned} \tag{55}$$

From the relations (8), the antisymmetric tensor $\mathscr{S}^{\mu\nu}$ and the dual tensor $\bar{\mathscr{S}}^{\mu\nu}$ are, to first order, given by

$$\begin{aligned}\mathscr{S}^{(1)\mu\nu}&=\frac{1}{p_n}\epsilon^{\mu\nu\alpha\beta}p_\alpha\mathscr{M}^{(1)}_\beta+\frac{1}{2p_n}(n_\nu\bar{\nabla}_\mu-n_\mu\bar{\nabla}_\nu)\mathscr{F}^{(0)},\\\tilde{\mathscr{S}}^{(1)\mu\nu}&=\frac{1}{p_n}(\mathscr{M}^{(1)\mu}p^\nu-\mathscr{M}^{(1)\nu}p^\mu)+\frac{1}{2p_n}\bar{\epsilon}^{\mu\nu\rho}\nabla_\rho\mathscr{F}^{(0)}.\end{aligned} \tag{56}$$

From the constraint (49), $\mathscr{M}^{(1)\mu}$ can be decomposed into components parallel and orthogonal to the momentum $\bar{p}^\mu$:

$$\mathscr{M}^{(1)\mu}=\mathscr{M}^{(1)\mu}_{\parallel}+\mathscr{M}^{(1)\mu}_{\perp}, \tag{57}$$

with the transverse constraint $\bar{p}_\mu\mathscr{M}^{(1)\mu}_{\perp}=0$. From the constraint (49) and the expression 53, the parallel and orthogonal parts are given, respectively, by

$$\begin{aligned}\mathscr{M}^{(1)\mu}_{\parallel}&=p_n\mathcal{M}^{(1)\mu}_{\parallel}\delta(p^2-m^2)-m\bar{p}^\mu\frac{\bar{p}\cdot B}{\bar{p}^2}\sum_s\mathcal{J}^{(0)}_{sn}\delta'(p^2-m^2),\\\mathscr{M}^{(1)\mu}_{\perp}&=p_n\mathcal{M}^{(1)\mu}_{\perp}\delta(p^2-m^2)-mB^\mu_\perp\sum_s\mathcal{J}^{(0)}_{sn}\delta'(p^2-m^2),\end{aligned} \tag{58}$$

with the relations

$$\mathcal{M}^{(1)\mu}_{\parallel}=\frac{m}{\bar{p}^2}\bar{p}^\mu\sum_s s\mathcal{J}^{(1)}_{sn}, \tag{59}$$

$$\bar{p}_\mu\mathcal{M}^{(1)\mu}_{\perp}\delta(p^2-m^2)=0. \tag{60}$$

Under this decomposition, we obtain the GCKE for the transverse distribution function $\mathscr{M}^{(0)\mu}_{\perp}$:

$$\begin{aligned}p^\nu\nabla_\nu\left(\frac{\mathscr{M}^{(0)\mu}_{\perp}}{p_n}\right)&=-\frac{mp_n}{\bar{p}^2}E^\mu_\perp\sum_s s\frac{\mathscr{J}^{(0)}_{sn}}{p_n}\\&\quad-\left(\frac{p_n\bar{p}^\mu}{\bar{p}^2}E^\nu+\bar{\epsilon}^{\mu\nu\alpha}B_\alpha\right)\frac{\mathscr{M}^{(0)}_{\perp\nu}}{p_n}.\end{aligned} \tag{61}$$

Note that we define the transverse electric and magnetic fields relative to the momentum $\bar{p}$.

$$E^\mu_\perp\equiv E^\mu-\frac{\bar{p}^\mu}{\bar{p}^2}E\cdot p,\quad B^\mu_\perp\equiv B^\mu-\frac{\bar{p}^\mu}{\bar{p}^2}B\cdot p.$$

C. Second-order equations

To obtain the GCKE for the first-order distribution functions $\mathscr{J}^{(1)}_{sn}$ and $\mathscr{M}^{(1)}_\mu$, we require the second-order Wigner equations. Fortunately, it suffices to consider only the following relevant equations:

$$\nabla_\mu\mathscr{J}^{(1)\mu}_s=-ms\mathscr{P}^{(2)}, \tag{62}$$

$$p_n\mathscr{P}^{(2)}+\frac{1}{2}n_\mu\nabla_\nu\tilde{\mathscr{S}}^{(1)\mu\nu}=0, \tag{63}$$

$$\bar{p}^\mu\mathscr{P}^{(2)}+\frac{1}{2}\Delta^\mu{}_\lambda\nabla_\nu\tilde{\mathscr{S}}^{(1)\lambda\nu}=0. \tag{64}$$

Substituting Eq. (63) into Eq. (62) and using Eqs. (45), (50), and (51) yields

$$\begin{aligned}p^\mu\nabla_\mu\left(\frac{\mathscr{J}^{(1)}_{sn}}{p_n}\right)&=-\frac{s}{4p_n}\bar{\epsilon}^{\mu\rho\sigma}(\partial^x_\lambda F_{\mu\rho})\partial^\lambda_p\mathscr{J}^{(0)}_{s\sigma}\\&\quad+\frac{ms}{2p_n^2}E_\nu\mathscr{M}^{(1)\nu}-\frac{s}{2p_n^2}\bar{\epsilon}^{\mu\rho\sigma}E_\mu\nabla_\rho\mathscr{J}^{(0)}_{s\sigma},\end{aligned} \tag{65}$$

which is the GCKE for $\mathscr{J}^{(1)}_{sn}$. Substituting Eq.(63) into Eq.(64) and using either Eq.(48) or Eq.(56) yields

$$\begin{aligned}p^\nu\nabla_\nu\left(\frac{\mathscr{M}^{(1)\mu}}{p_n}\right)&=\left(\frac{\bar{p}^\mu}{p_n}E^\nu-\bar{\epsilon}^{\mu\nu\alpha}B_\alpha\right)\frac{\mathscr{M}^{(1)}_\nu}{p_n}\\&\quad-\frac{1}{2p_n^2}\bar{\epsilon}^{\mu\nu\rho}E_\nu\nabla_\rho\mathscr{F}^{(0)}\\&\quad-\frac{1}{4p_n}\bar{\epsilon}^{\mu\nu\rho}(\partial^x_\lambda F_{\nu\rho})\partial^\lambda_p\mathscr{F}^{(0)}.\end{aligned} \tag{66}$$

This is the GCKE for $\mathscr{M}^{(1)\mu}$, from which the GCKE for the transverse part at first order can be obtained.

$$p^\nu\nabla_\nu\left(\frac{\mathscr{M}_\perp^{(1)\mu}}{p_n}\right)=-\frac{mp_n}{\bar{p}^2}E_\perp^\mu\sum_s s\frac{\mathscr{J}_{sn}^{(1)}}{p_n}-\left(\frac{p_n}{\bar{p}^2}\bar{p}^\mu E^\nu+\bar{\epsilon}^{\mu\nu\alpha}B_\alpha\right)\frac{\mathscr{M}_{\perp\nu}^{(1)}}{p_n}+\frac{p_\sigma}{2\bar{p}^2}(\bar{p}^\nu\bar{\epsilon}^{\mu\rho\sigma}-\bar{p}^\rho\bar{\epsilon}^{\mu\nu\sigma})\nabla_\nu\left(\frac{1}{p_n}\nabla_\rho\mathscr{F}^{(0)}\right).\tag{67}$$

Let us now summarize the full set of elements in the GCKT up to first order. The zeroth-order GCKEs are presented in Eqs. (50) and (51), subject to the constraint (17) and with the explicit forms given by (25) and (26). The remaining zeroth-order Wigner functions are determined through relations (21)-(24) and (27). Alternatively, the zeroth-order GCKEs may be expressed equivalently as Eqs. (50) and (61), using expressions (25) and (58), while the other Wigner functions follow from relations (21)-(24) and (27), together with (31). At first order, the GCKEs are given by Eqs. (65) and (66), constrained by (49) and with the functional forms specified in (53) and (55). The corresponding first-order Wigner functions are obtained from relations (45)-(48) and (56). An equivalent formulation of the first-order GCKEs is provided by Eqs. (65) and (67), along with expressions (53) and (58). In this case, the remaining first-order Wigner functions are derived from relations (45)-(48) and (56), in combination with (59).

IV. FRAME DEPENDENCE IN THE GCKT

In the GCKT, the basic distribution functions $\mathscr{J}_{sn}$ and $\mathscr{M}^\mu$ are defined with respect to the auxiliary timelike vector n^μ and consequently depend on this timelike vector n^μ. Since the normalized timelike vector n^μ can be identified as the four-velocity of a reference frame, it is important to discuss how these distribution functions transform between different frames. We assume that the distribution functions are defined in two different frames, n_μ and n'_μ. Then the Wigner functions $\mathscr{J}_s^\mu$ and $\mathscr{S}^{\mu\nu}$ (or $\tilde{\mathscr{S}}^{\mu\nu}$) should not depend on the choice of n_μ or n'_μ. This condition determines the frame dependence of the distribution functions.

At zeroth order, the independence of the Wigner functions $\mathscr{J}_s^{(0)\mu}$ from the choice of n_μ or n'_μ, together with the expression (21), yields

$$\frac{p^\mu\mathscr{J}_{sn}^{(0)}}{p_n}-\frac{sm\mathscr{M}_n^{(0)\mu}}{2p_n}=\frac{p^\mu\mathscr{J}_{sn'}^{(0)}}{p_{n'}}-\frac{sm\mathscr{M}_{n'}^{(0)\mu}}{2p_{n'}},$$

where the subscript n or n' has been attached to specify the vector with respect to which the magnetic-moment distribution is defined. The independence of the Wigner functions $\tilde{\mathscr{S}}^{(0)\mu\nu}$ from the choice of n_μ or n'_μ, together with the expression in Eq. (27), yields

$$\frac{\mathscr{M}_n^{(0)\mu}p^\nu}{p_n}-\frac{\mathscr{M}_n^{(0)\nu}p^\mu}{p_n}=\frac{\mathscr{M}_{n'}^{(0)\mu}p^\nu}{p_{n'}}-\frac{\mathscr{M}_{n'}^{(0)\nu}p^\mu}{p_{n'}}.$$

Contracting both sides of the two equations above with a four-vector η^μ, and assuming $\eta\cdot p\neq 0$, we can divide both sides by $\eta\cdot p$ and obtain, respectively,

$$\frac{\mathscr{J}_{sn}^{(0)}}{p_n}-\frac{sm(\eta\cdot\mathscr{M}_n^{(0)})}{2p_n(\eta\cdot p)}=\frac{\mathscr{J}_{sn'}^{(0)}}{p_{n'}}-\frac{sm(\eta\cdot\mathscr{M}_{n'}^{(0)})}{2p_{n'}(\eta\cdot p)},$$
$$\frac{\mathscr{M}_n^{(0)\mu}}{p_n}-\frac{p^\mu(\eta\cdot\mathscr{M}_n^{(0)})}{p_n(\eta\cdot p)}=\frac{\mathscr{M}_{n'}^{(0)\mu}}{p_{n'}}-\frac{p^\mu(\eta\cdot\mathscr{M}_{n'}^{(0)})}{p_{n'}(\eta\cdot p)}.$$

These relations express the frame dependence of the zeroth-order distribution functions $\mathscr{J}_{sn}^{(0)}$ and $\mathscr{M}_n^{(0)\mu}$ in the GCKT.

At first order, the independence of the first-order Wigner functions $\mathscr{J}_s^{(1)\mu}$ and $\tilde{\mathscr{S}}^{(1)\mu\nu}$ from the choice of n_μ or n'_μ, together with the expressions (45) and (56), leads to, respectively,

$$\begin{aligned}&\frac{p^\mu\mathscr{J}_{sn}^{(1)}}{p_n}-\frac{sm}{2p_n}\mathscr{M}_n^{(1)\mu}+\frac{s\epsilon^{\mu\nu\rho\sigma}n_\nu}{2p_n}\nabla_\rho\mathscr{J}_{s\sigma}^{(0)}\\=&\frac{p^\mu\mathscr{J}_{sn'}^{(1)}}{p_{n'}}-\frac{sm}{2p_{n'}}\mathscr{M}_{n'}^{(1)\mu}+\frac{s\epsilon^{\mu\nu\rho\sigma}n'_\nu}{2p_{n'}}\nabla_\rho\mathscr{J}_{s\sigma}^{(0)},\\&\frac{\mathscr{M}_n^{(1)\mu}p^\nu}{p_n}-\frac{\mathscr{M}_n^{(1)\nu}p^\mu}{p_n}+\frac{\epsilon^{\mu\sigma\nu\rho}n_\sigma}{2p_n}\nabla_\rho\mathscr{F}^{(0)}\\=&\frac{\mathscr{M}_{n'}^{(1)\mu}p^\nu}{p_{n'}}-\frac{\mathscr{M}_{n'}^{(1)\nu}p^\mu}{p_{n'}}+\frac{\epsilon^{\mu\sigma\nu\rho}n'_\sigma}{2p_{n'}}\nabla_\rho\mathscr{F}^{(0)}.\end{aligned}$$

As in the zeroth-order case, contracting both sides with η^μ and dividing by $\eta\cdot p$ leads to the frame dependence of the first-order distribution functions $\mathscr{J}_{sn}^{(1)}$ and $\mathscr{M}_n^{(1)\mu}$ in the GCKT:

$$\begin{aligned}&\frac{\mathscr{J}_{sn}^{(1)}}{p_n}-\frac{sm(\eta\cdot\mathscr{M}_n^{(1)})}{2p_n(\eta\cdot p)}+\frac{s\epsilon^{\mu\nu\rho\sigma}\eta_\mu n_\nu}{2p_n(\eta\cdot p)}\nabla_\rho\mathscr{J}_{s\sigma}^{(0)}\\=&\frac{\mathscr{J}_{sn'}^{(1)}}{p_{n'}}-\frac{sm(\eta\cdot\mathscr{M}_{n'}^{(1)})}{2p_{n'}(\eta\cdot p)}+\frac{s\epsilon^{\mu\nu\rho\sigma}\eta_\mu n'_\nu}{2p_{n'}(\eta\cdot p)}\nabla_\rho\mathscr{J}_{s\sigma}^{(0)},\\&\frac{\mathscr{M}_n^{(1)\mu}}{p_n}-\frac{p^\mu(\eta\cdot\mathscr{M}_n^{(1)})}{p_n(\eta\cdot p)}+\frac{\epsilon^{\mu\sigma\nu\rho}\eta_\nu n_\sigma}{2p_n(\eta\cdot p)}\nabla_\rho\mathscr{F}^{(0)}\\=&\frac{\mathscr{M}_{n'}^{(1)\mu}}{p_{n'}}-\frac{p^\mu(\eta\cdot\mathscr{M}_{n'}^{(1)})}{p_{n'}(\eta\cdot p)}+\frac{\epsilon^{\mu\sigma\nu\rho}\eta_\nu n'_\sigma}{2p_{n'}(\eta\cdot p)}\nabla_\rho\mathscr{F}^{(0)}.\end{aligned}$$

It might seem confusing that, to discuss the frame dependence associated with the auxiliary vector n^μ, we have introduced another auxiliary vector η^μ. In fact, we can avoid this complexity by choosing $\eta^\mu = p^\mu$, $\eta^\mu = n^\mu$, or $\eta^\mu = n'^\mu$. When we choose $\eta^\mu = p^\mu$, the p^2 term contributes m^2 to the denominator and leads to a nontrivial chiral limit at $m = 0$, which would undermine the advantage of the GCKT. We therefore choose an arbitrary vector η^μ, which is treated symmetrically with respect to both n^μ and n'^μ. When we choose $\eta^\mu = n^\mu$, the zeroth-order transformation rules for the frame dependence are given by

$$\frac{\mathscr{J}^{(0)}_{sn'}}{p_{n'}} - \frac{\mathscr{J}^{(0)}_{sn}}{p_n} = \frac{sm}{2p_{n'}p_n} n \cdot \mathscr{M}^{(0)}_{n'}, \tag{68}$$

$$\frac{\mathscr{M}^{(0)\mu}_{n'}}{p_{n'}} - \frac{\mathscr{M}^{(0)\mu}_{n}}{p_n} = \frac{p^\mu}{p_{n'}p_n} n \cdot \mathscr{M}^{(0)}_{n'}, \tag{69}$$

and the first-order transformation rules are given as follows:

$$\frac{\mathscr{J}^{(1)}_{sn'}}{p_{n'}} - \frac{\mathscr{J}^{(1)}_{sn}}{p_n} = \frac{sm}{2p_{n'}p_n} n \cdot \mathscr{M}^{(1)}_{n'} - \frac{s\epsilon^{\mu\nu\rho\sigma} n_\mu n'_\nu}{2p_{n'}p_n} \nabla_\rho \mathscr{J}^{(0)}_{s\sigma}, \tag{70}$$

$$\frac{\mathscr{M}^{(1)\mu}_{n'}}{p_{n'}} - \frac{\mathscr{M}^{(1)\mu}_{n}}{p_n} = \frac{p^\mu}{p_{n'}p_n} n \cdot \mathscr{M}^{(1)}_{n'} - \frac{\epsilon^{\mu\sigma\nu\rho} n_\nu n'_\sigma}{2p_{n'}p_n} \nabla_\rho \mathscr{F}^{(0)}. \tag{71}$$

The last term on the right-hand side of Eq. (70) is precisely the side-jump term in the CKT. In the GCKT, the first term on the right-hand side of Eq. (70) is an additional term due to finite mass. Such additional terms already exist in Eq. (68) at zeroth order. We can regard the last term on the right-hand side of Eq. (71) as the side-jump term for the distribution function $\mathscr{M}^{(1)\mu}$. It should be noted that, although the auxiliary vector n_μ was assumed to be constant from the beginning, the transformation rules between n_μ and n'_μ given in this section remain valid for an arbitrary n^μ that may depend on spacetime, since we have not applied any derivative operator to n_μ or n'_μ. This point will be exploited to derive the GCKT with a varying vector and to determine the Wigner functions in global equilibrium.

V. WIGNER FUNCTIONS IN GLOBAL EQUILIBRIUM

In this section, we apply the results given in Section III to find the solutions for the Wigner functions in global equilibrium in the presence of vorticity and EM fields. The zeroth-order Wigner function is obtained from free quantum field theory and is taken as a given input. For an unpolarized system, the fundamental distribution functions in GCKT read

$$\mathcal{J}^{(0)}_{sn} = \frac{1}{4\pi^3}\left[\frac{\theta(p_0 - m)}{1 + e^{\beta\cdot p - \bar{\mu}}} + \frac{\theta(-p_0 - m)}{1 + e^{-\beta\cdot p + \bar{\mu}}}\right], \tag{72}$$

$$\mathcal{M}^{(0)\mu}_{\perp} = 0, \tag{73}$$

where $\beta^\mu = u^\mu/T$ and $\bar{\mu} = \mu/T$, with temperature T, fluid velocity u^μ, and chemical potential μ. Note that, while we keep the chirality index s, the distribution function $\mathcal{J}^{(0)}_{sn}$ has no dependence on chirality in an unpolarized system. These expressions are the specific solutions of the GCKT at zeroth order if the following constraints in global equilibrium are satisfied under a varying $F^{\mu\nu}$:

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0, \quad \partial_\mu\bar{\mu} + F_{\mu\nu}\beta^\nu = 0. \tag{74}$$

The first constraint implies that the thermal vorticity (The definition differs from the conventional one by a minus sign here)

$$\Omega_{\mu\nu} = \frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu).$$

is a constant tensor, i.e., $\partial_\rho\Omega_{\mu\nu} = 0$, in global equilibrium [47]. For the second constraint, applying the partial derivative ∂_ν to both sides and using the commutativity of partial derivatives leads to the following integrability condition:

$$F_{\mu\lambda}\Omega_\nu^{\ \lambda} - F_{\nu\lambda}\Omega_\mu^{\ \lambda} = -\beta^\lambda(\partial_\lambda F_{\mu\nu}),$$

Contracting both sides with $\epsilon^{\alpha\beta\mu\nu}/2$ yields

$$\begin{aligned} F^\alpha{}_\lambda\tilde{\Omega}^{\beta\lambda} - F^\beta{}_\lambda\tilde{\Omega}^{\alpha\lambda} &= \tilde{F}^\alpha{}_\lambda\Omega^{\beta\lambda} - \tilde{F}^\beta{}_\lambda\Omega^{\alpha\lambda} \\ &= -\beta^\lambda(\partial_\lambda\tilde{F}^{\alpha\beta}). \end{aligned} \tag{75}$$

where we introduce the dual tensor $\tilde{\Omega}^{\alpha\beta} \equiv \epsilon^{\alpha\beta\rho\sigma}\Omega_{\rho\sigma}/2$ associated with the vorticity tensor. The derivative terms survive only for varying EM fields and vanish for constant ones. As will be demonstrated, a global equilibrium solution can be uniquely determined only when varying EM fields are present --- a condition not met in the constant-field case.

We determine the first-order solutions in global equilibrium by first using the frame dependence, followed by the kinetic equations.

A. Determination by frame dependence

Substituting the specific zeroth-order results (72) and (73), together with the first-order expressions (53) and (55), into the transformation rules (70) and (71), we obtain

$$\begin{aligned}
&\left(\mathcal{J}^{(1)}_{sn'}-\mathcal{J}^{(1)}_{sn}\right)\delta(p^2-m^2)\\
&-\frac{s}{2}\epsilon^{\mu\nu\alpha\beta}p_\mu\left(\frac{n'_\nu}{p_{n'}}-\frac{n_\nu}{p_n}\right)F_{\alpha\beta}\mathcal{J}^{(0)}_{sn}\delta'(p^2-m^2)\\
=&\frac{sm}{2p_n}n\cdot\mathcal{M}^{(1)}_{n'}\delta(p^2-m^2)\\
&-\frac{sm^2\epsilon^{\mu\nu\alpha\beta}n_\mu n'_\nu}{2p_{n'}p_n}F_{\alpha\beta}\mathcal{J}^{(0)}_{sn}\delta'(p^2-m^2)\\
&-\frac{s\epsilon^{\mu\nu\rho\sigma}n_\mu n'_\nu}{2p_{n'}p_n}\nabla_\rho\left[p_\sigma\mathcal{J}^{(0)}_{sn}\delta\left(p^2-m^2\right)\right],
\end{aligned}$$

$$\begin{aligned}
&\left(\mathcal{M}^{(1)\mu}_{n'}-\mathcal{M}^{(1)\mu}_{n}\right)\delta(p^2-m^2)\\
&-m\epsilon^{\mu\nu\alpha\beta}\left(\frac{n'_\nu}{p_{n'}}-\frac{n_\nu}{p_n}\right)F_{\alpha\beta}\mathcal{J}^{(0)}_{sn}\delta'(p^2-m^2)\\
=&\frac{p^\mu}{p_n}n\cdot\mathcal{M}^{(1)}_{n'}\delta(p^2-m^2)\\
&-\frac{mp^\mu\epsilon^{\lambda\nu\alpha\beta}n_\lambda n'_\nu}{p_{n'}p_n}F_{\alpha\beta}\mathcal{J}^{(0)}_{sn}\delta'(p^2-m^2)\\
&-\frac{\epsilon^{\mu\sigma\nu\rho}n_\nu n'_\sigma}{p_{n'}p_n}\nabla_\rho\left[m\mathcal{J}^{(0)}_{sn}\delta\left(p^2-m^2\right)\right]
\end{aligned}$$

Under the global equilibrium constraints in (74), we can verify the following results.

$$\nabla_\rho\mathcal{J}^{(0)}_{sn}=\mathcal{J}^{(0)\prime}_{sn}\Omega_{\rho\lambda}p^\lambda,\quad \mathcal{J}^{(0)\prime}_{sn}=\frac{\partial\mathcal{J}^{(0)}_{sn}}{\partial(\beta\cdot p)}$$

Together with the identity

$$p^\lambda\epsilon^{\mu\nu\rho\sigma}+p^\mu\epsilon^{\nu\rho\sigma\lambda}+p^\nu\epsilon^{\rho\sigma\lambda\mu}+p^\rho\epsilon^{\sigma\lambda\mu\nu}+p^\sigma\epsilon^{\lambda\mu\nu\rho}=0$$

the transformation rules can reduce to

$$\begin{aligned}
&\left(\mathcal{J}^{(1)}_{sn'}-\mathcal{J}^{(1)}_{sn}\right)\delta(p^2-m^2)\\
=&\frac{sm}{2p_n}n_\mu\left(\mathcal{M}^{(1)\mu}_{n'}-\frac{m}{p_{n'}}\tilde{\Omega}^{\mu\lambda}n'_\lambda\mathcal{J}^{(0)\prime}_{sn}\right)\delta\left(p^2-m^2\right)\\
&-\frac{s}{2}\left(\frac{n'_\mu}{p_{n'}}-\frac{n_\mu}{p_n}\right)\tilde{\Omega}^{\mu\lambda}p_\lambda\mathcal{J}^{(0)\prime}_{sn}\delta\left(p^2-m^2\right),\\
&\left(\mathcal{M}^{(1)\mu}_{n'}-\mathcal{M}^{(1)\mu}_{n}\right)\delta(p^2-m^2)\\
=&\frac{p^\mu}{p_n}n_\nu\left(\mathcal{M}^{(1)\nu}_{n'}-\frac{m}{p_{n'}}\tilde{\Omega}^{\nu\lambda}n'_\lambda\mathcal{J}^{(0)\prime}_{sn}\right)\delta\left(p^2-m^2\right)\\
&+m\left(\frac{n'_\nu}{p_{n'}}-\frac{n_\nu}{p_n}\right)\tilde{\Omega}^{\mu\nu}\mathcal{J}^{(0)\prime}_{sn}\delta\left(p^2-m^2\right)
\end{aligned}$$

It is evident that, if we redefine the distribution functions,

$$\begin{aligned}
\mathcal{J}^{(1)}_{sn}=&J^{(1)}_{sn}+\frac{s}{2p_n}\tilde{\Omega}^{\nu\lambda}p_\nu n_\lambda\mathcal{J}^{(0)\prime}_{sn},\\
\mathcal{M}^{(1)\mu}_{n}=&M^{(1)\mu}_{n}+\frac{m}{p_n}\tilde{\Omega}^{\mu\lambda}n_\lambda\mathcal{J}^{(0)\prime}_{sn},
\end{aligned}$$

Then the new distribution functions $J^{(1)}_{sn}$ and $M^{(1)\mu}_{n}$ transform as

$$\left(J^{(1)}_{sn'}-J^{(1)}_{sn}\right)\delta(p^2-m^2)=\frac{sm}{2p_n}n\cdot M^{(1)}_{n'}\delta(p^2-m^2)\tag{76}$$

$$\left(M^{(1)\mu}_{n'}-M^{(1)\mu}_{n}\right)\delta(p^2-m^2)=\frac{p^\mu}{p_n}n\cdot M^{(1)}_{n'}\delta(p^2-m^2)\tag{77}$$

These are identical to the zeroth-order transformation rules (68) and (69). Taking the trivial solution $J^{(1)}_s=0$ and $M^{(1)\mu}_n=0$, which satisfies these transformation rules, we obtain the following specific first-order solutions:

$$\begin{aligned}
\mathcal{J}^{(1)}_{sn}=&\frac{s}{2p_n}\tilde{\Omega}^{\nu\lambda}p_\nu n_\lambda\mathcal{J}^{(0)\prime}_{sn}\\
\mathcal{M}^{(1)\mu}_{n}=&\frac{m}{p_n}\tilde{\Omega}^{\mu\lambda}n_\lambda\mathcal{J}^{(0)\prime}_{sn}
\end{aligned}\tag{78}$$

In the following section, we will demonstrate that these solutions are fully consistent with the GCKT. In Section VI, we will show that they are, in fact, unique.

B. Determination by kinetic equations

In this section, we determine the first-order solution from the first-order kinetic equations. As given in Eqs. (53) and (58), the first-order GCKEs take the form:

$$\begin{aligned}
&p^\mu\nabla_\mu\left[\mathcal{J}^{(1)}_{sn}\delta(p^2-m^2)-\frac{s(B\cdot\bar{p})}{p_n}\mathcal{J}^{(0)}_{sn}\delta'(p^2-m^2)\right]\\
=&\frac{ms}{2p_n}E_\mu\left\{\frac{m\bar{p}^\mu}{\bar{p}^2}\sum_{s'}\left[s'\mathcal{J}^{(1)}_{s'n}\delta(p^2-m^2)\right.\right.\\
&\left.-\frac{(B\cdot\bar{p})}{p_n}\mathcal{J}^{(0)}_{s'n}\delta'(p^2-m^2)\right]+\mathcal{M}^{(1)\mu}_{\perp}\delta(p^2-m^2)\\
&\left.-\frac{m}{p_n}B^\mu_\perp\sum_{s'}\mathcal{J}^{(0)}_{s'n}\delta'(p^2-m^2)\right\}\\
&-\frac{s}{2}\bar{\epsilon}^{\mu\rho\sigma}\nabla_\mu\left\{\frac{1}{p_n}\nabla_\rho\left[p_\sigma\mathcal{J}^{(0)}_{sn}\delta(p^2-m^2)\right]\right\},
\end{aligned}\tag{79}$$

$$\begin{aligned}
&p^\nu\nabla_\nu\left[\mathcal{M}^{(1)\mu}_{\perp}\delta(p^2-m^2)-\frac{mB^\mu_\perp}{p_n}\sum_{s'}\mathcal{J}^{(0)}_{s'n}\delta'(p^2-m^2)\right]\\
=&-\left(\frac{p_n}{\bar{p}^2}\bar{p}^\mu E^\nu+\bar{\epsilon}^{\mu\nu\alpha}B_\alpha\right)\left[\mathcal{M}^{(1)}_{\perp\nu}\delta(p^2-m^2)\right.\\
&\left.-\frac{m}{p_n}B_{\perp\nu}\sum_{s'}\mathcal{J}^{(0)}_{s'n}\delta'(p^2-m^2)\right]-\frac{mp_n}{\bar{p}^2}E^\mu_\perp\sum_{s'}\\
&\left[s'\mathcal{J}^{(1)}_{s'n}\delta(p^2-m^2)-\frac{(B\cdot\bar{p})}{p_n}\mathcal{J}^{(0)}_{s'n}\delta'(p^2-m^2)\right]
\end{aligned}$$

$$+\frac{m}{2\bar{p}^2}\left(\bar{p}^{\nu}\bar{\epsilon}^{\mu\rho\sigma}-\bar{p}^{\rho}\bar{\epsilon}^{\mu\nu\sigma}\right)p_{\sigma}\nabla_{\nu}\left\{\frac{1}{p_n}\nabla_{\rho}\sum_{s}\left[\mathcal{J}^{(0)}_{sn}\delta(p^2-m^2)\right]\right\}. \tag{80}$$

Let us first consider the GCKE for $\mathcal{J}^{(1)}_{sn}$. Moving the second term in the square brackets in the first line of Eq.(79) to the right-hand side yields

$$\begin{aligned}
&p^{\mu}\nabla_{\mu}\left[\mathcal{J}^{(1)}_{sn}\delta(p^2-m^2)\right]\\
=&\frac{ms}{2p_n}E_{\mu}\left[\frac{m\bar{p}^{\mu}}{\bar{p}^2}\sum_{s'}s'\mathcal{J}^{(1)}_{s'n}+\mathcal{M}^{(1)\mu}_{\perp}\right]\delta(p^2-m^2)\\
&-\frac{m^2 s}{2p_n^2}(E\cdot B)\sum_{s'}\mathcal{J}^{(0)}_{s'n}\delta'(p^2-m^2)\\
&-\frac{s}{2p_n^2}\bar{\epsilon}^{\mu\rho\sigma}E_{\mu}\nabla_{\rho}\left[p_{\sigma}\mathcal{J}^{(0)}_{sn}\delta(p^2-m^2)\right]\\
&+p^{\mu}\nabla_{\mu}\left[\frac{s(B\cdot\bar{p})}{p_n}\mathcal{J}^{(0)}_{sn}\delta'(p^2-m^2)\right]\\
&-\frac{s}{2p_n}(\partial^x_{\lambda}B^{\sigma})\partial^{\lambda}_{p}\left[p_{\sigma}\mathcal{J}^{(0)}_{sn}\delta(p^2-m^2)\right].
\end{aligned}$$

Acting with the operator ∇_{μ} on the distribution functions and using Maxwell's equation $\partial_{\mu}\tilde{F}^{\mu\nu}=0$, we obtain

$$\begin{aligned}
&p^{\mu}\nabla_{\mu}\left[\mathcal{J}^{(1)}_{sn}\delta(p^2-m^2)\right]\\
=&\frac{ms}{2p_n}E_{\mu}\left(\frac{m\bar{p}^{\mu}}{\bar{p}^2}\sum_{s'}s'\mathcal{J}^{(1)}_{s'n}+\mathcal{M}^{(1)\mu}_{\perp}\right)\delta(p^2-m^2)\\
&+\frac{s}{2p_n^2}E_{\mu}\left(\bar{p}^{\mu}\tilde{\Omega}^{\nu\lambda}\bar{p}_{\nu}n_{\lambda}-\bar{p}^2\tilde{\Omega}^{\mu\lambda}n_{\lambda}\right.\\
&\quad\left.+p_n\tilde{\Omega}^{\mu\nu}\bar{p}_{\nu}\right)\mathcal{J}^{(0)\prime}_{sn}\delta(p^2-m^2)\\
&-\frac{s}{2p_n}\left[\beta^{\lambda}\partial^x_{\lambda}(B\cdot p)\right]\mathcal{J}^{(0)\prime}_{sn}\delta(p^2-m^2).
\end{aligned}$$

Using the constraint equation (75), we obtain

$$\beta^{\lambda}\partial_{\lambda}B_{\mu}=-\bar{\epsilon}_{\mu\nu\alpha}B^{\alpha}\tilde{\Omega}^{\nu\rho}n_{\rho}-E^{\nu}\Delta^{\rho}_{\mu}\tilde{\Omega}_{\rho\nu}. \tag{81}$$

Together with the decomposition

$$\nabla_{\mu}=\partial^x_{\mu}-E_{\mu}\partial_{p_n}+n_{\mu}E^{\nu}\bar{\partial}^p_{\nu}-\bar{\epsilon}_{\mu\nu\rho}B^{\nu}\bar{\partial}^{\rho}_{p},$$

we can write the kinetic equation as

$$\begin{aligned}
&p^{\mu}\nabla_{\mu}\left[\mathcal{J}^{(1)}_{sn}\delta(p^2-m^2)\right]\\
=&\frac{ms}{2p_n}E_{\mu}\left[\frac{m\bar{p}^{\mu}}{\bar{p}^2}\sum_{s'}s'\mathcal{J}^{(1)}_{s'n}+\mathcal{M}^{(1)\mu}_{\perp}\right]\delta(p^2-m^2)\\
&+\frac{s}{2p_n^2}E_{\mu}\left(\bar{p}^{\mu}\tilde{\Omega}^{\nu\lambda}\bar{p}_{\nu}-\bar{p}^2\tilde{\Omega}^{\mu\lambda}\right)n_{\lambda}\mathcal{J}^{(0)\prime}_{sn}\delta(p^2-m^2)\\
&+\frac{s}{2p_n}\bar{\epsilon}_{\mu\nu\alpha}\bar{p}^{\mu}B^{\alpha}\tilde{\Omega}^{\nu\lambda}n_{\lambda}\mathcal{J}^{(0)\prime}_{sn}\delta(p^2-m^2).
\end{aligned}$$

Using the identity

$$\begin{aligned}
&p^{\mu}\nabla_{\mu}\left[\frac{s}{2p_n}\tilde{\Omega}^{\nu\lambda}p_{\nu}n_{\lambda}\mathcal{J}^{(0)\prime}_{sn}\delta(p^2-m^2)\right]\\
=&\frac{sE_{\mu}}{2p_n^2}\left(\bar{p}^{\mu}\tilde{\Omega}^{\nu\lambda}\bar{p}_{\nu}+p_n^2\tilde{\Omega}^{\mu\lambda}\right)n_{\lambda}\mathcal{J}^{(0)\prime}_{sn}\delta(p^2-m^2)\\
&+\frac{s}{2p_n}\bar{\epsilon}_{\mu\nu\alpha}\bar{p}^{\mu}B^{\alpha}\tilde{\Omega}^{\nu\lambda}n_{\lambda}\mathcal{J}^{(0)\prime}_{sn}\delta(p^2-m^2),
\end{aligned}$$

With a little rearrangement, we obtain

$$\begin{aligned}
&p^{\mu}\nabla_{\mu}\left[\left(\mathcal{J}^{(1)}_{sn}-\delta\mathcal{J}^{(1)}_{sn}\right)\delta(p^2-m^2)\right]\\
=&\frac{ms}{2p_n}E_{\mu}\left[\frac{m\bar{p}^{\mu}}{\bar{p}^2}\sum_{s'}s'\left(\mathcal{J}^{(1)}_{s'n}-\delta\mathcal{J}^{(1)}_{sn}\right)\right]\delta(p^2-m^2)\\
&+\frac{ms}{2p_n}E_{\mu}\left[\mathcal{M}^{(1)\mu}_{\perp}-\delta\mathcal{M}^{(1)\mu}_{\perp}\right]\delta(p^2-m^2),
\end{aligned}$$

where we have defined

$$\delta\mathcal{J}^{(1)}_{sn}\equiv\frac{s}{2p_n}\tilde{\Omega}^{\nu\lambda}p_{\nu}n_{\lambda}\mathcal{J}^{(0)\prime}_{sn}, \tag{82}$$

$$\delta\mathcal{M}^{(1)\mu}_{\perp}\equiv\frac{m}{2p_n}\left(\tilde{\Omega}^{\mu\lambda}n_{\lambda}-\frac{\bar{p}^{\mu}}{\bar{p}^2}\tilde{\Omega}^{\nu\lambda}\bar{p}_{\nu}n_{\lambda}\right)\sum_{s}\mathcal{J}^{(0)\prime}_{sn}. \tag{83}$$

Similarly, let us consider the GCKE for $\mathcal{M}^{(1)\mu}_{\perp}$. Moving the second term inside the square brackets in the first line of Eq.(80) to the right-hand side of the equation yields

$$\begin{aligned}
&p^{\nu}\nabla_{\nu}\left[\mathcal{M}^{(1)\mu}_{\perp}\delta(p^2-m^2)\right]\\
=&-\left(\frac{p_n}{\bar{p}^2}\bar{p}^{\mu}E^{\nu}+\bar{\epsilon}^{\mu\nu\alpha}B_{\alpha}\right)\mathcal{M}^{(1)}_{\perp\nu}\delta(p^2-m^2)\\
&-\frac{mp_n}{\bar{p}^2}E^{\mu}_{\perp}\sum_{s'}s'\mathcal{J}^{(1)}_{s'n}\delta(p^2-m^2)\\
&-\frac{m}{2p_n}E^{\nu}\left(\frac{\bar{p}^{\mu}\bar{p}^{\rho}}{\bar{p}^2}\tilde{\Omega}_{\nu\rho}+\Delta^{\mu\lambda}\tilde{\Omega}_{\lambda\nu}\right)\sum_{s'}\mathcal{J}^{(0)\prime}_{sn}\delta(p^2-m^2)\\
&-\frac{mE\cdot\bar{p}}{2p_n^2}\left(\frac{\bar{p}^{\mu}\bar{p}_{\nu}}{\bar{p}^2}\tilde{\Omega}^{\nu\rho}n_{\rho}-\tilde{\Omega}^{\mu\nu}n_{\nu}\right)\sum_{s'}\mathcal{J}^{(0)\prime}_{sn}\delta(p^2-m^2)\\
&+\frac{mp_{\sigma}\beta^{\lambda}}{2\bar{p}^2p_n}\left(\bar{p}^{\mu}\partial_{\lambda}B^{\sigma}-\bar{p}^{\sigma}\partial_{\lambda}B^{\mu}\right)\sum_{s}\mathcal{J}^{(0)\prime}_{sn}\delta(p^2-m^2).
\end{aligned}$$

From the constraint equation (81), we obtain

$$
\begin{aligned}
& p^{\nu}\nabla_{\nu}\left[\mathcal{M}_{\perp}^{(1)\mu}\delta(p^2-m^2)\right] \\
= & -(\frac{p_n}{\bar{p}^2}\bar{p}^{\mu}E^{\nu}+\bar{\epsilon}^{\mu\nu\alpha}B_{\alpha})\mathcal{M}_{\perp\nu}^{(1)}\delta(p^2-m^2) \\
& -\frac{mp_n}{\bar{p}^2}E_{\perp}^{\mu}\sum_{s'}s'\mathcal{J}_{s'n}^{(1)}\delta(p^2-m^2) \\
& -\frac{mE\cdot\bar{p}}{2p_n^2}(\frac{\bar{p}^{\mu}\bar{p}_{\rho}}{\bar{p}^2}\tilde{\Omega}^{\rho\nu}-\tilde{\Omega}^{\mu\nu})n_{\nu}\sum_{s'}\mathcal{J}_{sn}^{(0)\prime}\delta(p^2-m^2) \\
& +\frac{mB_{\alpha}}{2p_n}(\bar{\epsilon}^{\mu\nu\alpha}-\frac{\bar{p}^{\mu}\bar{p}_{\sigma}}{\bar{p}^2}\bar{\epsilon}^{\sigma\nu\alpha})\tilde{\Omega}_{\nu\rho}n^{\rho}\sum_{s}\mathcal{J}_{sn}^{(0)\prime}\delta(p^2-m^2).
\end{aligned}
$$

Using the identity

$$
\begin{aligned}
& p^{\nu}\nabla_{\nu}\left[\frac{mn_{\lambda}}{2p_n}(\tilde{\Omega}^{\mu\lambda}-\frac{\bar{p}^{\mu}}{\bar{p}^2}\tilde{\Omega}^{\rho\lambda}\bar{p}_{\rho})\sum_{s}\mathcal{J}_{sn}^{(0)\prime}\delta(p^2-m^2)\right] \\
= & \frac{m}{2p_n^2}(E\cdot p)\tilde{\Omega}^{\mu\lambda}n_{\lambda}\sum_{s}\mathcal{J}_{sn}^{(0)\prime}\delta(p^2-m^2) \\
& -\frac{m}{2\bar{p}^2}\left(E^{\mu}\bar{p}_{\rho}+\bar{p}^{\mu}E_{\rho}\right)\tilde{\Omega}^{\rho\lambda}n_{\lambda}\sum_{s}\mathcal{J}_{sn}^{(0)\prime}\delta(p^2-m^2) \\
& +\frac{m}{2p_n\bar{p}^2}\bar{\epsilon}^{\nu\alpha\mu}\bar{p}_{\nu}B_{\alpha}\tilde{\Omega}^{\rho\lambda}\bar{p}_{\rho}n_{\lambda}\sum_{s}\mathcal{J}_{sn}^{(0)\prime}\delta(p^2-m^2) \\
& +\frac{m}{2p_n\bar{p}^2}\bar{p}^{\mu}\bar{\epsilon}^{\nu\alpha\rho}\bar{p}_{\nu}B_{\alpha}\tilde{\Omega}_{\rho\lambda}n^{\lambda}\sum_{s}\mathcal{J}_{sn}^{(0)\prime}\delta(p^2-m^2),
\end{aligned}
$$

we can write the equation as

$$
\begin{aligned}
& p^{\nu}\nabla_{\nu}\left[\left(\mathcal{M}_{\perp}^{(1)\mu}-\delta\mathcal{M}_{\perp}^{(1)\mu}\right)\delta(p^2-m^2)\right] \\
= & -\left(\frac{p_n}{\bar{p}^2}\bar{p}^{\mu}E_{\perp}^{\nu}+\bar{\epsilon}^{\mu\nu\alpha}B_{\alpha}\right)\left[\mathcal{M}_{\perp\nu}^{(1)}-\delta\mathcal{M}_{\perp\nu}^{(1)}\right]\delta(p^2-m^2) \\
& -\frac{mp_n}{\bar{p}^2}E_{\perp}^{\mu}\sum_{s}s\left(\mathcal{J}_{sn}^{(1)}-\delta\mathcal{J}_{sn}^{(1)}\right)\delta(p^2-m^2).
\end{aligned}
$$

We decompose the distribution functions as follows:

$$
\mathcal{J}_{sn}^{(1)}=J_{sn}^{(1)}+\delta\mathcal{J}_{sn}^{(1)}, \tag{84}
$$

$$
\mathcal{M}_{\perp}^{(1)\mu}=M_{\perp}^{(1)\mu}+\delta\mathcal{M}_{\perp}^{(1)\mu}, \tag{85}
$$

We can obtain the kinetic equations for $J_{sn}^{(1)}$ and $M_{\perp}^{(1)\mu}$.

$$
\begin{aligned}
& p^{\mu}\nabla_{\mu}\left[J_{sn}^{(1)}\delta(p^2-m^2)\right] \\
= & \frac{msE_{\mu}}{2p_n}(\frac{m\bar{p}^{\mu}}{\bar{p}^2}\sum_{s'}s'J_{s'n}^{(1)}+M_{\perp}^{(1)\mu})\delta(p^2-m^2),
\end{aligned} \tag{86}
$$

$$
\begin{aligned}
& p^{\nu}\nabla_{\nu}\left[M_{\perp}^{(1)\mu}\delta(p^2-m^2)\right] \\
= & -(\frac{p_n}{\bar{p}^2}\bar{p}^{\mu}E_{\perp}^{\nu}+\bar{\epsilon}^{\mu\nu\alpha}B_{\alpha})M_{\perp\nu}^{(1)}\delta(p^2-m^2)
\end{aligned}
$$

$$
-\frac{mp_n}{\bar{p}^2}E_{\perp}^{\mu}\sum_{s}sJ_{sn}^{(1)}\delta(p^2-m^2). \tag{87}
$$

If we choose the trivial solution

$$
J_{sn}^{(1)}=0,\quad M_{\perp}^{(1)\mu}=0, \tag{88}
$$

We obtain the specific solutions for $\mathcal{J}_{sn}^{(1)}$ and $\mathcal{M}_{\perp}^{(1)\mu}$.

$$
\begin{aligned}
\mathcal{J}_{sn}^{(1)} &= \frac{s}{2p_n}\tilde{\Omega}^{\nu\lambda}p_{\nu}n_{\lambda}\mathcal{J}_{sn}^{(0)\prime}, \\
\mathcal{M}_{\perp}^{(1)\mu} &= \frac{mn_{\lambda}}{2p_n\bar{p}^2}\left(\bar{p}^2\tilde{\Omega}^{\mu\lambda}-\bar{p}^{\mu}\tilde{\Omega}^{\nu\lambda}\bar{p}_{\nu}\right)\sum_{s}\mathcal{J}_{sn}^{(0)\prime}.
\end{aligned} \tag{89}
$$

These results are fully consistent with those derived from frame dependence, as shown in (78). It is remarkable that the results in (78), obtained from frame dependence, automatically satisfy the GCKT. Determining the solutions from frame dependence is considerably simpler than deriving them directly from the kinetic equations.

It follows that all other Wigner functions can be calculated directly:

$$
\begin{aligned}
\mathscr{J}_{s}^{(1)\mu} &= -\frac{s}{2}\tilde{\Omega}^{\mu\nu}p_{\nu}\mathcal{J}_{sn}^{(0)\prime}\delta(p^2-m^2) \\
&\quad +s\tilde{F}^{\mu\nu}p_{\nu}\mathcal{J}_{sn}^{(0)}\delta'(p^2-m^2), \\
\mathscr{M}^{(1)\mu} &= \frac{m}{2}\tilde{\Omega}^{\mu\lambda}n_{\lambda}\sum_{s}\mathcal{J}_{sn}^{(0)\prime} \\
&\quad -m\tilde{F}^{\mu\lambda}n_{\lambda}\sum_{s}\mathcal{J}_{sn}^{(0)}\delta'(p^2-m^2), \\
\mathscr{S}^{(1)\mu\nu} &= m\Omega^{\mu\nu}\mathcal{J}_{sn}^{(0)\prime}\delta(p^2-m^2) \\
&\quad -2mF^{\mu\nu}\mathcal{J}_{sn}^{(0)}\delta'(p^2-m^2), \\
\tilde{\mathscr{S}}^{(1)\mu\nu} &= m\tilde{\Omega}^{\mu\nu}\mathcal{J}_{sn}^{(0)\prime}\delta(p^2-m^2) \\
&\quad -2m\tilde{F}^{\mu\nu}\mathcal{J}_{sn}^{(0)}\delta'(p^2-m^2), \\
\mathscr{P}^{(1)} &= 0, \\
\mathscr{F}^{(1)} &= 0.
\end{aligned}
$$

We note that the Wigner functions $\mathscr{J}_{s}^{(1)\mu}$, $\mathscr{S}^{(1)\mu\nu}$, and $\tilde{\mathscr{S}}^{(1)\mu\nu}$ are all independent of the auxiliary vector n^{μ}, as expected.

VI. GENERAL SOLUTION

In the previous section, we chose the specific solution (88) or (89). It is important to investigate whether other solutions are possible and what the general solution is. To this end, we must express the functions $J_{sn}^{(1)}$ and $M_{\perp}^{(1)\mu}$ in terms of all possible vectors or tensors involved in the problem, such as n^{μ}, u^{μ}, $F^{\mu\nu}$, $\Omega^{\mu\nu}$, and their derivatives. Thus, it would be convenient to identify n^μ with u^μ. However, we cannot naively replace n^μ with u^μ because n^μ has been assumed to be a constant vector from the outset, whereas the fluid velocity u^μ varies in space and time. We can circumvent this issue by exploiting the frame dependence of the distribution functions, since the transformation rules are valid for arbitrarily varying vectors n^μ and u^μ, as mentioned at the end of Section IV. These transformation rules can then be used to obtain the relations between the distributions defined with respect to the constant n^μ and the varying u^μ. Since the Wigner equations for constant n^μ have already been derived, the equations for varying u^μ can be obtained from these relations.

We can directly use the transformation rules (76) and (77) and set $n'^\mu = u^\mu$:

$$\begin{aligned}&J^{(1)}_{su}\delta(p^2-m^2)\\&=J^{(1)}_{sn}\delta(p^2-m^2)-\frac{sm(u\cdot M^{(1)}_n)}{2p_u}\delta(p^2-m^2),\end{aligned}\tag{90}$$

$$\begin{aligned}&M^{(1)\mu}_u\delta(p^2-m^2)\\&=M^{(1)\mu}_n\delta(p^2-m^2)-\frac{p^\mu(u\cdot M^{(1)}_n)}{p_u}\delta(p^2-m^2).\end{aligned}\tag{91}$$

Since the solutions (89) automatically satisfy the constraint (49), the distribution functions $J^{(1)}_{sn}$ and $M^{(1)\mu}_n$ must also satisfy this constraint. It follows that $M^{(1)\mu}_{u\|}$ is related to $J^{(1)}_{sn}$ and $M^{(1)\mu}_n$ by

$$\begin{aligned}M^{(1)\mu}_{u\|}&=\frac{m\bar{p}^\mu_u}{\bar{p}^2_u}\sum_s s\mathcal{J}^{(1)}_{su}\\&=\frac{m\bar{p}^\mu_u}{\bar{p}^2_u}\sum_s s\mathcal{J}^{(1)}_{sn}-\frac{m^2\bar{p}^\mu_u}{\bar{p}^2_u p_u}u\cdot M^{(1)}_n,\end{aligned}$$

which lead to the relations

$$\begin{aligned}J^{(1)}_{su}&=J^{(1)}_{sn}-\frac{sm}{2p_u}u\cdot M^{(1)}_{n\perp}\\&\quad-\frac{sm^2(p_u-p_n n\cdot u)}{2p_u\bar{p}^2}\sum_{s'}s'J^{(1)}_{s'n},\\M^{(1)\mu}_{u\perp}&=M^{(1)\mu}_{n\perp}+\frac{(p_u p^\mu-m^2u^\mu)}{p^2-p^2_u}u\cdot M^{(1)}_{n\perp}\\&\quad+\frac{mp_n}{\bar{p}^2}\left[\frac{(p_n-p_u n\cdot u)\bar{p}^\mu_u}{p^2-p^2_u}\right.\\&\quad\left.+(u^\mu n\cdot u-n^\mu)\right]\sum_s s\mathcal{J}^{(1)}_{sn}.\end{aligned}$$

For simplicity, the on-shell delta function $\delta(p^2-m^2)$ has been omitted from both sides of the above three equations. Then, using these relations together with the kinetic equations (86) and (87), it is straightforward to derive the following equations for $J^{(1)}_{su}$ and $M^{(1)\mu}_{u\perp}$:

$$\begin{aligned}&p^\mu\nabla_\mu\left[J^{(1)}_{su}\delta(p^2-m^2)\right]\\&=\frac{sm}{2p_u}\left(E^\mu-p^\nu\nabla_\nu u_\mu\right)\\&\quad\times\left(M^{(1)}_{\perp\mu}+\frac{m\bar{p}_\mu}{\bar{p}^2}\sum_{s'}s'J^{(1)}_{s'u}\right)\delta(p^2-m^2),\end{aligned}\tag{92}$$

$$\begin{aligned}&p^\nu\nabla_\nu\left[M^{(1)\mu}_{u\perp}\delta(p^2-m^2)\right]\\&=-\frac{mp_u}{\bar{p}^2}\left[E^\mu_\perp-\Delta^{\mu\nu}_\perp p^\lambda(\nabla_\lambda u_\nu)\right]\sum_s sJ^{(0)}_{su}\delta(p^2-m^2)\\&\quad-\left(\frac{p_u}{\bar{p}^2}\bar{p}^\mu E^\nu+\bar{\epsilon}^{\mu\nu\alpha}B_\alpha+\frac{\bar{p}^2u^\mu-p_u\bar{p}^\mu}{\bar{p}^2}p^\lambda\nabla_\lambda u^\nu\right)\\&\quad\times M^{(1)}_{u\perp\nu}\delta(p^2-m^2),\end{aligned}\tag{93}$$

where we define the transverse projector

$$\Delta^{\mu\nu}_\perp=\Delta^{\mu\nu}-\frac{1}{\bar{p}^2}\bar{p}^\mu\bar{p}^\nu.$$

In general, $J^{(1)}_{su}$ and $M^{(1)\mu}_{u\perp}$ can be expressed as

$$J^{(1)}_{su}=\frac{s}{p_u}\tilde{\Omega}^{\mu\nu}p_\mu u_\nu\mathcal{X}^J_\Omega+\frac{s}{p^3_u}\tilde{F}^{\mu\nu}p_\mu u_\nu\mathcal{X}^J_F,\tag{94}$$

$$\begin{aligned}M^{(1)\mu}_{u\perp}&=\frac{m}{p^2_u}\Delta^\mu_\lambda\tilde{\Omega}^{\lambda\nu}\bar{p}_\nu\tilde{\mathcal{X}}^M_\Omega+\frac{m}{p^4_u}\Delta^\mu_\lambda\tilde{F}^{\lambda\nu}\bar{p}_\nu\tilde{\mathcal{X}}^M_F\\&\quad+\frac{m}{p_u\bar{p}^2}\left(\bar{p}^2\tilde{\Omega}^{\mu\nu}-\bar{p}^\mu_u\tilde{\Omega}^{\lambda\nu}p_\lambda\right)u_\nu\mathcal{X}^M_\Omega\\&\quad+\frac{m}{p^3_u\bar{p}^2}\left(\bar{p}^2\tilde{F}^{\mu\nu}-\bar{p}^\mu_u\tilde{F}^{\lambda\nu}p_\lambda\right)u_\nu\mathcal{X}^M_F.\end{aligned}\tag{95}$$

Without loss of generality, we can assume that all $\mathcal{X}$ are functions of

$$z=\beta\cdot p-\bar{\mu},\ \tilde{z}=\beta\cdot p+\bar{\mu},\quad \bar{m}=m/T.$$

By substituting the expressions (94) and (95) into the kinetic equations (92) and (93) and requiring that the equations be satisfied, we obtain

$$\mathcal{X}^J_F=0,\quad \mathcal{X}^M_F=0,\quad \tilde{\mathcal{X}}^M_F=0.$$

The necessity of these conditions arises from an imbalance: the field-derivative terms on the left-hand side have no counterparts on the right-hand side with which to cancel. Note that the assumption of a varying EM field is essential for this simple conclusion, since the derivative terms automatically vanish for a constant EM field. Now let us determine the remaining terms associated with the vorticity tensor from the kinetic equations in global equilibrium.

$$\begin{aligned}
&p^\lambda\nabla_\lambda\left[\frac{s}{p_u}\tilde{\Omega}^{\mu\nu}p_\mu u_\nu\mathcal{X}^J_\Omega\delta(p^2-m^2)\right]\\
=&\frac{sm^2}{2p_u^2\bar{p}^2}\left[E_\mu-p^\lambda(\nabla_\lambda u_\mu)\right]\left[\left(\bar{p}^2\tilde{\Omega}^{\mu\nu}-\bar{p}^\mu\tilde{\Omega}^{\kappa\nu}p_\kappa\right)u_\nu\mathcal{X}^M_\Omega\right.\\
&\left.+2\bar{p}^\mu\tilde{\Omega}^{\lambda\nu}p_\lambda u_\nu\mathcal{X}^J_\Omega\right]\delta(p^2-m^2)\\
&+\frac{sm^2}{2p_u^3}\left[E_\mu-p^\lambda(\nabla_\lambda u_\mu)\right]\tilde{\Omega}^{\mu\nu}\bar{p}_\nu\tilde{\mathcal{X}}^M_\Omega\delta(p^2-m^2).
\end{aligned}$$

After applying the operator $p^\lambda\nabla_\lambda$ to the following terms, together with the decomposition

$$T\tilde{\Omega}_{\mu\nu}=\omega_\mu u_\nu-\omega_\nu u_\mu+\epsilon_{\mu\nu\rho\sigma}\varepsilon^\rho u^\sigma,$$

and combining like terms, we obtain

$$\begin{aligned}
0=&C_1(E\cdot p)(\omega\cdot p)\delta(p^2-m^2)\\
&+C_2(\omega\cdot E)\delta(p^2-m^2)\\
&+C_3(\varepsilon\cdot p)(\omega\cdot p)\delta(p^2-m^2)\\
&+C_4(\varepsilon\cdot\omega)\delta(p^2-m^2)\\
&+\frac{1}{p_uT}\bar{\epsilon}_{\lambda\mu\nu}p^\lambda\omega^\mu B^\nu\mathcal{X}^J_\Omega\delta(p^2-m^2)\\
&-\frac{m^2}{2p_u^3T}\bar{\epsilon}_{\lambda\mu\nu}\bar{p}^\lambda\varepsilon^\mu E^\nu\tilde{\mathcal{X}}^M_\Omega\delta(p^2-m^2),
\end{aligned}\tag{96}$$

where the coefficients C_1, C_2, C_3, and C_4 are given by

$$\begin{aligned}
C_1&=\frac{1}{p_u^2T}\left[\mathcal{X}^J_\Omega-\frac{2p_u}{T}\frac{\partial\mathcal{X}^J_\Omega}{\partial\tilde{z}}-\frac{m^2}{2\bar{p}^2}(2\mathcal{X}^J_\Omega-\mathcal{X}^M_\Omega)\right]\\
C_2&=\frac{1}{T}\left(\mathcal{X}^J_\Omega-\frac{m^2}{2p_u^2}\mathcal{X}^M_\Omega\right)\\
C_3&=\frac{m}{p_uT}\left[\frac{1}{T}\frac{\partial\mathcal{X}^J_\Omega}{\partial\bar{m}}-\frac{m}{2\bar{p}^2}(2\mathcal{X}^J_\Omega-\mathcal{X}^M_\Omega)+\frac{m}{2p_u^2}\tilde{\mathcal{X}}^M_\Omega\right]\\
C_4&=\frac{m^2T}{2p_u}\left(2\mathcal{X}^J_\Omega-\mathcal{X}^M_\Omega-\frac{\bar{p}^2}{p_u^2}\tilde{\mathcal{X}}^M_\Omega\right).
\end{aligned}$$

Given that the first four terms are linearly independent, we obtain the following constraints:

$$C_1=0,\quad C_2=0,\quad C_3=0,\quad C_4=0,$$

which lead to

$$\mathcal{X}^J_\Omega=\frac{m^2}{2p_u^2}\mathcal{X}^M_\Omega,\ \tilde{\mathcal{X}}^M_\Omega=\mathcal{X}^M_\Omega,\ \frac{\partial\mathcal{X}^J_\Omega}{\partial\tilde{z}}=0,\ \frac{\partial\mathcal{X}^J_\Omega}{\partial\bar{m}}=0.\tag{97}$$

Once these relations are satisfied, Eq. (96) becomes

$$\begin{aligned}
0=&\frac{1}{p_uT}\bar{\epsilon}_{\lambda\mu\nu}p^\lambda\omega^\mu B^\nu\mathcal{X}^J_\Omega\delta(p^2-m^2)\\
&-\frac{m^2}{2p_u^3T}\bar{\epsilon}_{\lambda\mu\nu}p^\lambda\varepsilon^\mu E^\nu\tilde{\mathcal{X}}^M_\Omega\delta(p^2-m^2)
\end{aligned}\tag{98}$$

Using the constraint (75)

$$\epsilon_{\mu\nu\rho\sigma}u^\nu(\varepsilon^\rho E^\sigma-\omega^\rho B^\sigma)=-u^\nu u^\lambda\partial_\lambda\tilde{F}_{\mu\nu}$$

Substituting it into Eq. (98), we obtain

$$\frac{1}{p_uT}u^\nu p^\mu u^\lambda(\partial_\lambda\tilde{F}_{\mu\nu})\mathcal{X}^J_\Omega\delta(p^2-m^2)=0\tag{99}$$

Evidently, for an arbitrarily varying EM field, the only solution satisfying Eqs. (97) and (99) is

$$\mathcal{X}^J_\Omega=0,\quad\mathcal{X}^M_\Omega=0,\quad\tilde{\mathcal{X}}^M_\Omega=0$$

These results demonstrate that the particular solution (89) is unique. We have therefore uniquely determined the GCKT under global equilibrium conditions. The essential point is the introduction of a varying EM field. Without it, Eq. (99) would be automatically satisfied for a constant field, thereby allowing all terms compatible with the relations (97). This explains the discrepancy between our previous results in [41] and those in [33], where a constant EM field was assumed. It is thus remarkable that the quantum kinetic equations determine the Wigner functions at global equilibrium; in particular, the presence of a varying EM field uniquely fixes their form up to first order in $\hbar$.

The uniqueness in the varying-field case arises because the gradient terms associated with the EM field introduce additional constraints that eliminate the zero modes of the kinetic operator. These constraints originate from the integrability conditions of global equilibrium, which involve derivatives of the electromagnetic field and are associated with nonzero modes. They imply a coupling between zero modes and nonzero modes, imposing additional restrictions that force the coefficients of the homogeneous solutions to vanish. Physically, this means that spacetime-dependent fields introduce additional ``source" terms that break the degeneracy of the kinetic operator, thereby leading to a unique solution.

VII. SUMMARY

In this work, we have systematically investigated the frame dependence of distribution functions within the GCKT. Motivated by the ambiguities associated with transforming distribution functions across different frames and the non-uniqueness of global equilibrium solutions reported in prior studies, we derived explicit transformation rules governing this frame dependence. By applying these rules and allowing for a varying EM field, we were able to uniquely determine the Wigner functions describing fermions of arbitrary mass in global equilibrium under the influence of vorticity and EM fields. By incorporating the previously neglected frame dependence, our results resolve the ambiguities encountered previously and establish a self-consistent theoretical framework for describing quantum transport phenomena. This framework thus provides a novel method for determining transport properties in global equilibrium directly from quantum kinetic theory.

Recently, several spin-related puzzles have emerged in the study of spin physics in heavy-ion collisions. Theoretical predictions in this context rely heavily on distribution functions. However, when spin degrees of freedom are taken into account, distribution functions may exhibit nontrivial frame dependence. Whether such frame dependence can affect theoretical predictions is therefore an important question. Our current work establishes a self-consistent theoretical framework that addresses this issue, providing a solid foundation for future applications in relativistic heavy-ion collisions and other related fields.

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