Equilibrium-Completion Entropy Potential: A Measure of Thermalization Capacity in Cosmology*

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Abstract: Entropy production in cosmological nonequilibrium processes can be difficult to characterize when intermediate states lack a well-defined local thermodynamic entropy. Motivated by monopole production and strongly supercooled cosmological phase transitions, we introduce the equilibrium-completion entropy potential (ECEP), a counterfactual measure of equilibrium entropy capacity. For a given underlying theory, the Landau-frame energy density of the actual local state defines a maximum-entropy equilibrium-completion state, and its equilibrium entropy density defines the ECEP. We derive an exact local identity that relates the divergence of the associated ECEP current to the mismatch between the actual stress tensor and the equilibrium-completion stress tensor. We present the corresponding finite-domain balance and resolved-hard matching formulations, with the caveat that predictive closure of the hard-region contribution lies beyond the present ECEP treatment. As cosmological benchmarks, we show that a strictly equilibrated crossover transition conserves comoving ECEP despite changes in the equation of state and in the number of relativistic degrees of freedom, whereas the equilibrium-completion target of a first-order phase transition generally includes the phase-coexistence branch.

Keywords: cosmological phase transition, nonequilibrium thermodynamics, entropy production

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I. Introduction

Cosmological phase transitions [1, 2] can have a wide range of important consequences, including the production of topological defects [3−5], stochastic gravitational waves [6−10], the baryon asymmetry [11], primordial magnetic fields [12], and curvature perturbations [13−16]. Particularly interesting are strongly supercooled first-order phase transitions, in which the Universe can remain trapped in a metastable phase for an extended period before the transition completes [17−19]. Such transitions may involve a substantial release of vacuum energy and can enhance the prospects for observable gravitational-wave signals [20]. At the same time, the associated entropy production can significantly affect cosmological relics [21]: it can dilute the final abundance of particles or topological defects and modify the redshifting of primordial gravitational waves. This raises an intriguing question near the boundary of successful transition completion: as a strongly supercooled transition is pushed toward this boundary from the completing side, can the final yield of topological defects be made arbitrarily small[1)], and how large a gravitational-wave signal can remain observable? Addressing such questions requires a controlled treatment of the entropy generated during the transition and of the resulting cosmological dilution.

Strongly supercooled first-order phase transitions [18, 26−35] can also enter regimes in which standard simplified descriptions are difficult to justify. Because nucleation may be delayed for a substantial fraction of a Hubble time, bubbles can grow to sizes comparable to the Hubble radius [17, 18]; in sufficiently prolonged transitions, even larger physical scales may become relevant. At the same time, the conversion of vacuum energy need not immediately produce a thermal bath: the intermediate state may contain strongly nonthermal particle distribu-

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1) This is especially relevant for the monopole problem [22−25] associated with (partially) unified gauge theories.

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tions, relativistic bulk motion, coherent scalar field configurations and waves, and other forms of stress that remain far from local thermodynamic equilibrium (LTE) [6, 31, 36]. At scales comparable to the bubble size, large spatial inhomogeneities can then limit the validity of a homogeneous Friedmann-Lemaître-Robertson-Walker (FLRW) description [37−39], while the absence of LTE precludes the usual assignment of a local thermodynamic entropy density. This creates a basic difficulty in entropy accounting. If the transition ultimately completes, thermalizes, and the Universe returns to an approximately homogeneous FLRW state, one generally expects reheating and relaxation to produce a substantial increase in comoving entropy relative to the pre-transition state. Yet the physical origin and spacetime accumulation of this entropy increase are embedded precisely in the highly inhomogeneous and strongly nonequilibrium stage where the conventional thermodynamic description is least reliable.

This situation motivates us to define a concept that can characterize the thermalization capacity of a nonequilibrium state, especially in curved spacetime, where a canonical conserved global matter energy is generally unavailable [40]. Since curved spacetime is locally flat, it is natural to begin with local quantities. The distinguished local frame is the Landau frame, in which the matter energy flux vanishes. In such a frame, all the energy can, in principle, be thermalized, and it is therefore reasonable to consider the maximum entropy associated with this counterfactual auxiliary thermalization process. In this work, we formulate this idea as the *equilibrium-completion entropy potential* (ECEP) and introduce an associated ECEP current to describe its spacetime evolution. Although the ECEP concept is motivated by strongly supercooled first-order phase transitions in which a homogeneous FLRW description may fail, we nevertheless recognize that, in such cases, the spacetime contribution to ECEP can be decomposed into what we term resolved and hard regions. The contribution from resolved regions can, in principle, be handled with predictive closure relations, whereas the treatment of the hard-region contribution remains an open problem, which we leave for future work.

The remainder of this paper is organized as follows. In Sec. II, we define the ECEP and discuss its physical interpretation and relation to the reference-state and local Gibbs constructions [41−45] in relativistic statistical mechanics. In Sec. III, we derive the local ECEP identity, Eq. (36), which is the central result of our paper. We also discuss the finite-domain balance relation and its further decomposition into contributions from resolved and hard regions. In Sec. IV, we illustrate the use of ECEP with cosmological phase-transition benchmarks, considering an equilibrium crossover in a homogeneous FLRW universe and a first-order phase transition involving phase coexistence. We summarize our results and discuss their implications and limitations in Sec. V. Throughout the main text, we restrict to the simplest setting, in which all degrees of freedom included in the construction can participate in the thermalization process and no conserved charge needs to be treated explicitly. In the terminology used in Appendix A, this corresponds to a single thermalization-connected sector with no additional conserved-charge or other completion constraints. More general cases are treated in Appendix A. Throughout this work, we use the metric signature $(-,+,+,+)$.

II. Equilibrium-Completion Entropy Potential

A. Motivation and Construction Principle

The need for a measure of thermalization capacity is related to a crucial difference between flat and curved spacetimes. For an isolated system of fixed volume described by a given underlying theory in Minkowski spacetime, once the total energy is known, the equilibrium entropy that the system can ultimately support is fixed by ordinary equilibrium thermodynamics, regardless of the complexity of the intermediate nonequilibrium evolution. In this case, introducing an additional measure of thermalization capacity is generally unnecessary. However, in a generic curved spacetime, there is no canonical globally conserved matter energy that can play the same role. It is therefore natural to characterize thermalization capacity first from a local perspective.

A local measure of the energy available for thermalization must therefore be identified. The energy density measured by an arbitrary observer is unsuitable for this purpose because it includes the kinetic energy associated solely with the overall motion of the local system relative to that observer. This observer-dependent boost contribution is not an intrinsic resource that can be converted into internal thermal energy through local relaxation. We therefore anchor the construction to the Landau frame [46] of the actual stress-energy tensor $T^{\mu\nu}$, defined, whenever it exists uniquely, by a future-directed unit timelike eigenvector u_L^μ satisfying[1)]

$$T^{\mu\nu}u_{L\nu} = -\rho_L u_L^\mu, \qquad u_L^\mu u_{L\mu} = -1. \tag{1}$$

The corresponding Landau energy density can equivalently be expressed as

$$\rho_L = T^{\mu\nu}u_{L\mu}u_{L\nu}. \tag{2}$$

1) Here $T^{\mu\nu}$ denotes the stress-energy tensor of the matter degrees of freedom included in the ECEP construction, obtained by varying their matter action $S_{\rm m}$ with respect to the spacetime metric, $T_{\mu\nu} = -(2/\sqrt{-g})\delta S_{\rm m}/\delta g^{\mu\nu}$ [40].

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It represents the energy density measured in the local energy frame comoving with u_L^μ, in which the energy flux vanishes. This prescription removes the energy associated solely with an overall boost of the local system relative to an external observer, while retaining its intrinsic Landau-frame energy content.

We note that the timelike Landau eigenvector associated with an exact vacuum stress tensor $T^{\mu\nu} = -\epsilon g^{\mu\nu}$ is not unique. For a vacuum-dominated state with a nonzero non-vacuum component, $T^{\mu\nu} = -\epsilon g^{\mu\nu} + T^{\mu\nu}_{\rm nv}$, this ambiguity is absent provided that $T^{\mu\nu}_{\rm nv}$ has a unique timelike Landau eigenvector. The vacuum term changes the corresponding eigenvalue but not the eigenvector, even when the non-vacuum component is very small. Realistic cosmological applications always contain a nonzero non-vacuum component, so this exact degeneracy does not arise in our applications. We therefore restrict the ECEP current to states for which the actual stress-energy tensor has a unique timelike Landau eigenvector.

To characterize the thermalization capacity of the actual local state, one may ask the counterfactual question: what is the maximum entropy density that can be supported for a given Landau-frame energy density? The use of constrained entropy maximization is consistent with the standard equilibrium principle [47, 48] of selecting the entropy-maximizing state compatible with the prescribed macroscopic constraint. One virtue of this construction is that it does not require choosing an equilibrium phase or knowing which state the actual system will reach in the future.

This maximum-entropy prescription is particularly important when the equilibrium thermodynamics admits a first-order phase transition. The maximization must then include the full set of admissible equilibrium configurations, including phase-coexistence states, rather than selecting a homogeneous phase by an additional rule. Under the standard assumptions of extensive thermodynamics and in the thermodynamic limit, the coexistence branch also makes the resulting equilibrium-completion entropy density a continuous and differentiable function of the energy density across the first-order coexistence interval.

With these construction principles established, we now introduce the concept of ECEP in two steps. We first define the equilibrium thermodynamic state functions of the counterfactual target in an auxiliary Minkowski system, and then evaluate them locally, in Sec. II.C, using the Landau-frame energy density of the actual curved-spacetime state.

B. Equilibrium-Completion Thermodynamics

Target-side thermodynamics is the standard thermodynamics of stable equilibrium in the underlying theory, formulated here as constrained entropy maximization. We first express this maximization in terms of *statistical states*. By a statistical state, we mean a probabilistic specification of the microscopic state of the system [49]: a density operator in a quantum description, or a probability distribution or measure on phase space in a classical description. We denote such a state generically by Γ and its statistical entropy by $S_{\rm stat}[\Gamma]$. Thus, in a quantum description with $k_B = 1$,

$$S_{\rm stat}[\Gamma] = -{\rm Tr}(\Gamma \ln \Gamma), \tag{3}$$

whereas, in a classical description, $S_{\rm stat}$ denotes the corresponding Gibbs entropy.

To make the construction precise, consider an auxiliary system in Minkowski spacetime with physical volume V and Hamiltonian H_V, and specify its rest-frame energy density ρ. We denote by

$$\mathcal{A}_V(\rho) \tag{4}$$

the admissible set which consists of normalized statistical states of this system that are compatible with the underlying theory and satisfy the prescribed energy constraint. In a quantum description, its elements satisfy

$${\rm Tr}\Gamma = 1, \qquad \frac{1}{V}{\rm Tr}(\Gamma H_V) = \rho, \tag{5}$$

with the corresponding phase-space normalization and energy average interpreted in the classical sense. At fixed V, an equilibrium-completion state is any admissible statistical state that maximizes $S_{\rm stat}$. When such a maximizer exists, it can be written as

$$\Gamma_{{\rm EC},V} \in \underset{\Gamma\in\mathcal{A}_V(\rho)}{\arg\max}\, S_{\rm stat}[\Gamma]. \tag{6}$$

We emphasize that this is a global maximization: if multiple local maxima exist, only the state with the highest entropy is selected, whereas metastable equilibrium states are discarded.

The corresponding equilibrium-completion entropy-density function is defined by the thermodynamic limit

$$s_{\rm EC}(\rho) = \lim_{V\to\infty} \frac{1}{V} \sup_{\Gamma\in\mathcal{A}_V(\rho)} S_{\rm stat}[\Gamma], \tag{7}$$

provided that this limit exists. The volume V appearing here is an auxiliary thermodynamic volume used to define bulk equilibrium thermodynamics. Thus, $s_{\rm EC}(\rho)$ is a thermodynamic state function of the prescribed energy density.

The standard intensive thermodynamic quantities of the equilibrium-completion target are obtained from this state function. Wherever $s_{\rm EC}(\rho)$ is differentiable, we

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define the equilibrium-completion temperature $T_{\mathrm{EC}}(\rho)$ by

$$\frac{1}{T_{\mathrm{EC}}} = \frac{ds_{\mathrm{EC}}}{d\rho}. \tag{8}$$

The equilibrium-completion pressure can be obtained from the Euler relation

$$p_{\mathrm{EC}} = T_{\mathrm{EC}} s_{\mathrm{EC}} - \rho. \tag{9}$$

Both T_{EC} and p_{EC} are thermodynamic properties of the counterfactual equilibrium target. They should not be identified with the temperature or thermodynamic pressure of an arbitrary actual nonequilibrium state.

The maximum-entropy construction has an important feature when the equilibrium thermodynamics includes a first-order phase transition. Suppose that two equilibrium phases coexist at temperature T_c and pressure p_c, with energy and entropy densities (ρ_1, s_1) and (ρ_2, s_2), where $\rho_1 < \rho_2$. For a prescribed intermediate energy density $\rho_1 < \rho < \rho_2$, the maximum-entropy target is represented, in the thermodynamic limit, by a phase-coexistence state, which can be written as

$$\rho = (1-\lambda)\rho_1 + \lambda\rho_2, \tag{10}$$

$$s_{\mathrm{EC}}(\rho) = (1-\lambda)s_1 + \lambda s_2, \tag{11}$$

where $0 \le \lambda \le 1$ denotes the equilibrium volume fraction of the second phase in the auxiliary thermodynamic system. Along this coexistence branch,

$$T_{\mathrm{EC}} = T_c, \qquad p_{\mathrm{EC}} = p_c. \tag{12}$$

The coexistence construction also explicitly demonstrates the regularity stated in Sec. II.A. Since each coexisting phase satisfies $\rho_i + p_c = T_c s_i$, one obtains

$$\frac{ds_{\mathrm{EC}}}{d\rho} = \frac{s_2 - s_1}{\rho_2 - \rho_1} = \frac{1}{T_c}. \tag{13}$$

The adjacent pure-phase equilibrium branches also satisfy $ds_{\mathrm{EC}}/d\rho = 1/T$ and approach $1/T_c$ at the endpoints of coexistence. Thus, $s_{\mathrm{EC}}(\rho)$ is continuous and differentiable throughout the coexistence interval, although higher-order derivatives are generally nonanalytic.

C. ECEP of an Actual Nonequilibrium State

We now return to the physical curved spacetime and combine the two components of the construction developed above. Let $(\mathcal{M}, g_{\mu\nu})$ be the physical spacetime, and consider a spacetime point x at which the stress-energy tensor has the unique future-directed timelike Landau eigenvector introduced in Sec. II.A. The state therefore determines the Landau-frame energy density $\rho_L(x)$, while Sec. II.B provides the equilibrium-completion thermodynamic functions. We define the local ECEP density by evaluating the equilibrium-completion entropy-density function at this local energy density,

$$s_{\mathrm{EC}}(x) \equiv s_{\mathrm{EC}}(\rho_L(x)). \tag{14}$$

We use the same symbol s_{EC} for the thermodynamic state function and for its value in the actual state; the arguments distinguish the two. Thus, $s_{\mathrm{EC}}(\rho)$ denotes the equilibrium thermodynamic state function defined in Sec. II.B, whereas $s_{\mathrm{EC}}(x)$ denotes a scalar field on the actual spacetime. Analogously, the equilibrium-completion temperature and pressure associated with the actual local energy density are

$$T_{\mathrm{EC}}(x) \equiv T_{\mathrm{EC}}(\rho_L(x)), \qquad p_{\mathrm{EC}}(x) \equiv p_{\mathrm{EC}}(\rho_L(x)). \tag{15}$$

These quantities characterize the counterfactual equilibrium target associated with the actual local state.

The ECEP assigned to an actual state also depends on the resolution at which its stress-energy tensor is specified. In the main text, we use the stress-energy tensor of the retained degrees of freedom at the adopted resolution and treat them as a single thermalization-connected sector without imposing additional conserved-charge constraints. The coarse-graining used in the first-order benchmark is specified in Sec. IV.B, while thermalization connectivity and conserved charges are discussed in Appendix A. Once the theory, sector partition, and retained constraints are fixed, the function $s_{\mathrm{EC}}(\rho)$ is fixed. Changing the resolution can still change ρ_L and, therefore, the ECEP assigned to the same physical state. Likewise, changing the completion timescale τ_{EC} changes the ECEP only if it changes the sector partition or the constraints retained in the completion problem.

The construction has the following consistency property. Suppose that the actual local state itself realizes the maximum-entropy equilibrium state at the same Landau energy density. In that case,

$$s_{\mathrm{EC}}(x) = s_{\mathrm{th}}(x), \tag{16}$$

where $s_{\mathrm{th}}(x)$ is the ordinary thermodynamic entropy density of the actual equilibrium state. LTE alone is insufficient for this equality to hold. For example, a locally equilibrated metastable phase need not realize the global entropy maximum at fixed energy density when a higher-entropy equilibrium configuration is available. The relev-

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ant condition is therefore that the actual state itself coincides with the equilibrium state selected by the maximum-entropy construction.

It is convenient to define the ECEP current using u_L^μ

$$J_{\rm EC}^\mu(x) \equiv s_{\rm EC}(x) u_L^\mu(x). \tag{17}$$

In general, $J_{\rm EC}^\mu$ should be distinguished from the entropy current of the actual nonequilibrium state, if such a notion exists. The divergence of $J_{\rm EC}^\mu$ encodes the evolution of the ECEP, which is analyzed in Sec. III.

The ECEP current provides a covariant definition of the ECEP for a finite spatial domain. Let Σ be a spacelike hypersurface in $(\mathcal{M}, g_{\mu\nu})$, let n^μ be its future-directed unit timelike normal satisfying $n^\mu n_\mu = -1$, and let $d\Sigma$ denote the proper spatial volume element induced on Σ. For a finite domain $D \subset \Sigma$, we define the finite-domain ECEP as the flux of the ECEP current through D,

$$P_{\rm EC}[D,\Sigma] \equiv -\int_D J_{\rm EC}^\mu n_\mu \, d\Sigma. \tag{18}$$

We introduce the relative Lorentz factor between the hypersurface-normal observers and the local Landau frame,

$$\gamma_{un} \equiv -u_L^\mu n_\mu \geq 1, \tag{19}$$

Equation (18) can equivalently be written as

$$P_{\rm EC}[D,\Sigma] = \int_D s_{\rm EC} \gamma_{un} \, d\Sigma. \tag{20}$$

When a hypersurface orthogonal to the Landau flow is available, such that $n^\mu = u_L^\mu$, then $\gamma_{un} = 1$, and the definition reduces to its rest-frame form

$$P_{\rm EC}[D,\Sigma] = \int_D s_{\rm EC} \, d\Sigma. \tag{21}$$

More generally, however, constructing the ECEP in the Landau frame does not require the existence of a foliation orthogonal to the Landau flow. Since $J_{\rm EC}^\mu$ is not assumed to be conserved, the finite-domain ECEP is generally associated with both the specified spatial domain and the hypersurface on which it is evaluated, and it need not be a conserved charge.

D. Relation to Reference-State and Local Gibbs Constructions

It is useful to compare the above construction with earlier equilibrium-reference constructions. In relativistic dissipative hydrodynamics, Ref. [41] introduces a fictitious local-equilibrium reference state whose energy density is matched to that of the actual fluid in the Landau frame. The entropy density of this reference state provides an equilibrium baseline for describing dissipative corrections and constructing the nonequilibrium entropy current. With the same equilibrium equation of state and constraints, this reference entropy density may therefore coincide with $s_{\rm EC}(\rho_L)$. A related construction is used in Ref. [42], where an equivalent equilibrium entropy density is assigned to a pre-equilibrium energy density and used as a matching variable when switching from a pre-equilibrium description to hydrodynamics.

The present construction differs both in how the reference state is selected and in the role assigned to its entropy density. The equilibrium-completion target is defined by global entropy maximization over the admissible states at fixed Landau-frame energy density. Thus, in the absence of additional completion constraints, metastable entropy extrema are not selected as completion states, and the phase-coexistence branch is included when it yields the stable equilibrium maximum. The resulting $s_{\rm EC}$ is not used as a local-equilibrium baseline for a dissipative expansion or as a matching variable between two dynamical descriptions. Instead, we interpret $s_{\rm EC}$ as the equilibrium entropy capacity associated with the actual local state and study its evolution through the ECEP current and finite-domain balance.

It is also useful to compare ECEP with the local Gibbs construction of nonequilibrium statistical mechanics [43–45]. Both constructions employ constrained entropy maximization with locally prescribed information and have a counterfactual character. The key difference is that the local Gibbs construction is intrinsically based on a chosen Cauchy hypersurface, whereas ECEP starts pointwise and is intrinsically connected to the Landau frame. Therefore, in the local Gibbs construction, one first obtains a global entropy associated with the Cauchy hypersurface; a local entropy density or current arises only through further localization. In the ECEP construction, the local ECEP entropy density is obtained first, and a finite-domain ECEP is then obtained by integration. In general, the Landau flow need not be hypersurface-orthogonal, so the relation between the local Gibbs construction and the local ECEP is not straightforward; we leave this question for future study.

III. The Local ECEP Identity and ECEP Evolution

A. Exact local ECEP identity

We derive the local evolution law for the ECEP current introduced in Eq. (17). We consider a spacetime region in which the Landau four-velocity u_L^μ is well defined and the stress-energy tensor entering the ECEP construction is locally conserved,

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$$\nabla_\mu T^{\mu\nu} = 0. \tag{22}$$

The equilibrium-completion state functions of Sec. II.B can be evaluated along a differentiable thermodynamic branch.

We first introduce the spatial projector orthogonal to the Landau four-velocity,

$$h^{\mu\nu} \equiv g^{\mu\nu} + u_L^\mu u_L^\nu, \tag{23}$$

which satisfies $h^{\mu\nu}u_{L\nu} = 0$. Because u_L^μ is the timelike eigenvector of $T^{\mu\nu}$, the energy flux vanishes in the Landau frame. Thus, the stress-energy tensor can be decomposed as [46]

$$T^{\mu\nu} = \rho_L u_L^\mu u_L^\nu + p_{\rm act} h^{\mu\nu} + \pi^{\mu\nu}. \tag{24}$$

Here, the isotropic component of the actual spatial stress, $p_{\rm act}$, and the anisotropic stress, $\pi^{\mu\nu}$, are given by

$$p_{\rm act} \equiv \frac{1}{3} h_{\mu\nu} T^{\mu\nu}, \qquad \pi^{\mu\nu} \equiv h^\mu{}_\alpha h^\nu{}_\beta T^{\alpha\beta} - p_{\rm act} h^{\mu\nu}, \tag{25}$$

so that

$$u_{L\mu}\pi^{\mu\nu} = 0, \qquad \pi^\mu{}_\mu = 0. \tag{26}$$

The local deformation of the Landau congruence enters through its expansion scalar

$$\theta \equiv \nabla_\mu u_L^\mu \tag{27}$$

and shear tensor

$$\sigma_{\mu\nu} \equiv h_\mu{}^\alpha h_\nu{}^\beta \nabla_{(\alpha} u_{L\beta)} - \frac{1}{3}\theta h_{\mu\nu}, \tag{28}$$

where parentheses denote symmetrization. The shear tensor is transverse and traceless: $u_L^\mu \sigma_{\mu\nu} = 0$ and $\sigma^\mu{}_\mu = 0$.

Projecting Eq. (22) along u_L^ν and using the decomposition (24) yields

$$0 = u_{L\nu}\nabla_\mu T^{\mu\nu} = -u_L^\mu \nabla_\mu \rho_L - (\rho_L + p_{\rm act})\theta - \pi^{\mu\nu}\nabla_\mu u_{L\nu}. \tag{29}$$

Because $\pi^{\mu\nu}$ is symmetric, spatial, and traceless, its contraction with $\nabla_\mu u_{L\nu}$ receives contributions only from the shear component of the velocity gradient. Therefore,

$$\pi^{\mu\nu}\nabla_\mu u_{L\nu} = \pi^{\mu\nu}\sigma_{\mu\nu}, \tag{30}$$

and Eq. (29) becomes the local energy equation in the Landau frame

$$u_L^\mu \nabla_\mu \rho_L + (\rho_L + p_{\rm act})\theta + \pi^{\mu\nu}\sigma_{\mu\nu} = 0. \tag{31}$$

We now evaluate the divergence of the ECEP current, $J_{\rm EC}^\mu = s_{\rm EC} u_L^\mu$. Using its definition,

$$\nabla_\mu J_{\rm EC}^\mu = u_L^\mu \nabla_\mu s_{\rm EC} + s_{\rm EC}\theta. \tag{32}$$

Because $s_{\rm EC}(x) = s_{\rm EC}(\rho_L(x))$, the thermodynamic relation in Eq. (8) yields

$$u_L^\mu \nabla_\mu s_{\rm EC} = \frac{1}{T_{\rm EC}} u_L^\mu \nabla_\mu \rho_L. \tag{33}$$

Using also the Euler relation,

$$T_{\rm EC} s_{\rm EC} = \rho_L + p_{\rm EC}, \tag{34}$$

Equation (32) can therefore be expressed as

$$\nabla_\mu J_{\rm EC}^\mu = \frac{1}{T_{\rm EC}} \left[u_L^\mu \nabla_\mu \rho_L + (\rho_L + p_{\rm EC})\theta \right]. \tag{35}$$

Finally, substituting the actual energy equation (31) yields

$$\nabla_\mu J_{\rm EC}^\mu = \frac{(p_{\rm EC} - p_{\rm act})\theta - \pi^{\mu\nu}\sigma_{\mu\nu}}{T_{\rm EC}}. \tag{36}$$

Eq. (36) is the exact local ECEP identity in the setting considered in the main text. Here, "exact" means that the relation follows algebraically from local stress-energy conservation and the equilibrium thermodynamic identities defining $s_{\rm EC}$, $T_{\rm EC}$, and $p_{\rm EC}$.

The identity applies wherever the quantities entering it are sufficiently regular and the same differentiable equilibrium-completion thermodynamics can be followed locally. The phase-coexistence branch discussed in Sec. II.B does not, in itself, obstruct this construction, because $s_{\rm EC}(\rho)$ is continuous and differentiable across the coexistence interval under the assumptions made there.

An equation with a similar algebraic form appears in the context of relativistic dissipative hydrodynamics. Ref. [41] introduces a local-equilibrium reference entropy current $S_0^\mu = s_0 u^\mu$ and derives its divergence in its Eq. (58), which, after appropriate conversion, takes the same form as Eq. (36) in the Landau-matched limit. As discussed in Sec. II.D, the reference state in Ref. [41] is used as the local-equilibrium baseline of dissipative hydrodynamics. Its entropy current S_0^μ forms the equilibrium part of the nonequilibrium entropy current, whose divergence is con-

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strained by the second law. In the present work, $J^{\mu}_{\rm EC}$ is instead used to characterize the equilibrium entropy capacity, which relies on global entropy maximization, and the ECEP evolution is not constrained by a corresponding second law. The derivation of Eq. (36) does not assume that the actual state is close to LTE or require hydrodynamic constitutive relations.

B. Physical interpretation of the local identity

The physical content of the local ECEP identity can be understood by comparing the actual Landau-frame energy response with a counterfactual adiabatic response of the corresponding equilibrium-completion state. At a given spacetime point, consider the equilibrium-completion state that has the same energy density ρ_L as the actual state. If this state were displaced infinitesimally along its equilibrium family while subject to the same instantaneous expansion scalar θ, and if its equilibrium entropy were transported adiabatically, its energy-density response would satisfy

$$u^{\mu}_L \nabla_{\mu}\rho\big|_{\rm EC,ad} = -(\rho_L + p_{\rm EC})\theta. \tag{37}$$

By contrast, the Landau-frame energy density satisfies Eq. (31),

$$u^{\mu}_L \nabla_{\mu}\rho_L = -(\rho_L + p_{\rm act})\theta - \pi^{\mu\nu}\sigma_{\mu\nu}. \tag{38}$$

Eq. (35) can therefore be expressed as

$$T_{\rm EC}\nabla_{\mu}J^{\mu}_{\rm EC} = u^{\mu}_L \nabla_{\mu}\rho_L - u^{\mu}_L \nabla_{\mu}\rho\big|_{\rm EC,ad}. \tag{39}$$

Thus, the local change in the ECEP measures the extent to which the actual Landau-frame energy-density response deviates from the adiabatic response that would preserve the comoving entropy of the corresponding equilibrium-completion target. The factor $1/T_{\rm EC} = ds_{\rm EC}/d\rho_L$ is therefore a thermodynamic response coefficient of that target.

Eq. (39) provides a direct interpretation of Eq. (36). The first term on the right-hand side of Eq. (36),

$$\frac{(p_{\rm EC} - p_{\rm act})\theta}{T_{\rm EC}}, \tag{40}$$

represents a mismatch in the isotropic response. At the same Landau energy density, $p_{\rm act}$ is the isotropic mechanical stress carried by the actual state, whereas $p_{\rm EC}$ is the thermodynamic pressure of the equilibrium-completion state. Their difference therefore quantifies how the isotropic component of the actual energy response differs from the adiabatic reference. For example, for an expanding Landau congruence with $\theta > 0$, and in the absence of the anisotropic contribution, $p_{\rm act} < p_{\rm EC}$ implies that the actual energy density decreases more slowly than in the equilibrium-adiabatic reference evolution, thereby making a positive contribution to $\nabla_{\mu}J^{\mu}_{\rm EC}$. This interpretation applies equally to states near and far from LTE.

The second term,

$$-\frac{\pi^{\mu\nu}\sigma_{\mu\nu}}{T_{\rm EC}}, \tag{41}$$

accounts for the part of the actual Landau-frame energy response associated with anisotropic stress and shear deformation of the Landau congruence. The equilibrium-completion state is locally isotropic and therefore has no counterpart to this term. The sign of the contraction $\pi^{\mu\nu}\sigma_{\mu\nu}$ is not fixed in a generic nonequilibrium state, so this term does not necessarily represent irreversible dissipation.

In general, these terms should not be interpreted as thermodynamic work. The Landau congruence is defined by the local energy flow and need not be a material congruence; diffusion or other transport relative to u^{μ}_L may occur [46], and in a general field-theoretic or multicomponent system, there need not be a unique material velocity. Moreover, $p_{\rm act}$ need not be a thermodynamic pressure, and the standard interpretation of $-pdV$ as thermodynamic work requires the process under consideration to be quasistatic, which need not be the case.

We note that, if the actual stress is close to the equilibrium-completion pressure and satisfies the first-order viscous relations [46, 50]

$$p_{\rm act} = p_{\rm EC} + \Pi, \qquad \Pi = -\zeta\theta, \tag{42}$$

and

$$\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}, \tag{43}$$

for $\zeta, \eta \geq 0$, Eq. (36) reduces to

$$\nabla_{\mu}J^{\mu}_{\rm EC} = \frac{\zeta\theta^2 + 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu}}{T_{\rm EC}} \geq 0. \tag{44}$$

This result resembles the standard formula for bulk- and shear-viscous entropy production, but its physical content differs from the law of entropy increase unless Eq. (16) can be established. In general, neither constitutive relation is available, and the sign of $\nabla_{\mu}J^{\mu}_{\rm EC}$ is unrestricted.

C. Worldtube ECEP balance

The local ECEP evolution equation can be integrated

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over a finite spacetime region to obtain a covariant balance relation for the finite-domain ECEP defined in Eq. (18). Consider a finite spacetime worldtube W bounded by an initial spacelike domain $D_i \subset \Sigma_i$, a final spacelike domain $D_f \subset \Sigma_f$, and a timelike side boundary $\mathcal{B}$. Let m^μ denote the outward-pointing unit spacelike normal to $\mathcal{B}$, and let $d\Sigma_3$ be the proper three-volume element induced on $\mathcal{B}$. We define the accumulated boundary ECEP flux by

$$\Phi_{\rm EC} \equiv \int_{\mathcal{B}} J^\mu_{\rm EC} m_\mu \, d\Sigma_3, \tag{45}$$

with the sign convention that $\Phi_{\rm EC} > 0$ denotes a net outward ECEP flux through the worldtube.

Applying the covariant divergence theorem to $J^\mu_{\rm EC}$, and choosing the orientations of the initial and final spacelike boundaries consistently with the finite-domain definition (18), gives

$$P_{\rm EC}[D_f,\Sigma_f] - P_{\rm EC}[D_i,\Sigma_i] = \int_W \nabla_\mu J^\mu_{\rm EC} \, dV_4 - \Phi_{\rm EC}, \tag{46}$$

where dV_4 is the proper spacetime volume element. This expression gives the general worldtube ECEP balance: the change in the finite-domain ECEP equals the integrated local ECEP response within the worldtube minus the net ECEP carried outward through its timelike boundary.

If the local ECEP identity (36) applies throughout W, the bulk term may be written explicitly as

$$\begin{aligned} &P_{\rm EC}[D_f,\Sigma_f] - P_{\rm EC}[D_i,\Sigma_i] \\ &= \int_W \frac{(p_{\rm EC} - p_{\rm act})\theta - \pi^{\mu\nu}\sigma_{\mu\nu}}{T_{\rm EC}} \, dV_4 - \Phi_{\rm EC}. \end{aligned} \tag{47}$$

Two suitable choices of domain eliminate the boundary term. First, since $J^\mu_{\rm EC} = s_{\rm EC} u^\mu_L$, the boundary flux vanishes pointwise if

$$u^\mu_L m_\mu = 0 \qquad \text{on } \mathcal{B}. \tag{48}$$

This condition is naturally realized by a Landau-comoving worldtube, whose timelike side boundary is generated by transporting an initial spatial boundary along the Landau flow. Second, in a fully periodic numerical domain, opposite faces are identified and the corresponding ECEP fluxes cancel pairwise, so the net boundary flux vanishes.

If the ECEP current is locally conserved, $\nabla_\mu J^\mu_{\rm EC} = 0$, then the finite-domain ECEP is conserved for any worldtube with vanishing boundary flux. In general, the sign of $\nabla_\mu J^\mu_{\rm EC}$ is not fixed. Therefore, the finite-domain ECEP is neither a conserved charge nor associated with any principle of monotonic increase.

D. Resolved-hard decomposition and matching

The worldtube balance derived above is, in principle, an exact relation. However, if insufficient closure relations are available for the components of the stress-energy tensor, its predictive power is severely limited. With this in mind, we describe a spacetime region as resolved if the evolution of the stress-energy tensor in that region can be predictively closed. For example, predictive closure may be provided by controlled hydrodynamic constitutive relations, kinetic equations [51], or directly evolved field equations. Conversely, we describe a spacetime region as hard if the adopted description does not provide a controlled closure for the subsequent evolution of the stress-energy tensor[1)]. The resolved–hard classification is not necessarily determined by whether the states involved are close to LTE. A strongly nonequilibrium configuration can still be resolved if sufficiently complete dynamical equations are solved.

For strongly supercooled first-order cosmological phase transitions, this distinction has a natural realization. Within a reduced description, the approximately homogeneous false-vacuum region is typically resolved, whereas the bubble-wall neighborhood and strongly nonequilibrium regions behind the wall are natural candidates for the hard region. In these regions, coherent field configurations, relativistic flows, and nonthermal particle distributions may invalidate the adopted closure. This assignment is not intrinsic: with sufficiently complete field-theoretic or kinetic dynamics and adequate resolution, the same regions may instead become resolved.

For bookkeeping in a finite domain, a hard region can be excised and represented by data on a surrounding matching surface placed in a finite collar where the resolved description has already become reliable again. If the hard set consists of mutually disjoint connected components H_a, let $\mathcal{M}_a$ be the corresponding enclosing matching hypersurfaces lying in such resolved collars. We define the integrated hard-region ECEP response by

$$\mathcal{K}_{\rm EC}[H_a;\mathcal{M}_a] \equiv \int_{\mathcal{M}_a} J^\mu_{\rm EC} \, d\Sigma_\mu, \tag{49}$$

with the orientation chosen to be consistent with the worldtube balance. Equation (46) then takes the form of the resolved-hard decomposition:

1) We note that ECEP remains well defined in a hard region. If a more complete calculation supplies the actual $T^{\mu\nu}(x)$ inside a hard region and the local conditions of Secs. 2 and 3.1 are satisfied, the ECEP and the local identity can still be evaluated there.

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$$P_{\rm EC}[D_f,\Sigma_f]-P_{\rm EC}[D_i,\Sigma_i] = \int_{\Omega_{\rm res}} \nabla_\mu J^\mu_{\rm EC}\, dV_4 + \sum_a \mathcal{K}_{\rm EC}[H_a;\mathcal{M}_a] - \Phi_{\rm EC}, \tag{50}$$

where $\Omega_{\rm res}$ denotes the resolved spacetime region within the worldtube. The components of the computation that require additional dynamical information are thus incorporated into the hard-region ECEP response, for which case-by-case studies may be necessary for different types of typical nonequilibrium processes.

IV. Cosmological Phase-Transition Benchmarks

A. Crossover Phase Transitions

We first consider a crossover transition in a spatially homogeneous and isotropic FLRW universe as a simple benchmark for the ECEP construction[1)]. We work in the ideal equilibrium limit, in which the local state continuously follows the stable equilibrium branch as the Universe expands. A cosmological comoving observer then coincides with the Landau flow. The expansion scalar is $\theta = 3H$, and exact isotropy implies $\sigma_{\mu\nu}=0$ and $\pi^{\mu\nu}=0$. The local ECEP identity, Eq. (36), therefore reduces to

$$u^\mu_L \nabla_\mu s_{\rm EC} + 3H s_{\rm EC} = \frac{3H}{T_{\rm EC}}\left(p_{\rm EC}-p_{\rm act}\right). \tag{51}$$

In a homogeneous FLRW background, this relation can equivalently be expressed in terms of derivatives with respect to cosmic time t, where a denotes the scale factor,

$$\frac{d}{dt}\left(a^3 s_{\rm EC}\right) = a^3 \frac{3H}{T_{\rm EC}}\left(p_{\rm EC}-p_{\rm act}\right). \tag{52}$$

The above equation already shows that ECEP generation is associated with the mismatch between the actual mechanical stress and the stress in the equilibrium-completion state, rather than solely with a changing equilibrium equation of state.

In an equilibrium crossover, the thermodynamic state evolves smoothly along a single stable equilibrium branch,

$$T \longmapsto \left\{\rho_{\rm eq}(T),\, p_{\rm eq}(T),\, s_{\rm eq}(T),\, \bar{\phi}(T)\right\}, \tag{53}$$

where $\bar{\phi}(T)$ denotes a possible equilibrium value of the order parameter at temperature T. Because the actual local state is assumed to coincide at all times with the stable equilibrium state selected by the maximum-entropy construction,

$$p_{\rm act} = p_{\rm EC} = p_{\rm eq}, \qquad T_{\rm EC} = T. \tag{54}$$

The local ECEP identity then gives

$$u^\mu_L \nabla_\mu s_{\rm EC} + 3H s_{\rm EC} = 0, \tag{55}$$

or

$$\frac{d}{dt}\left(a^3 s_{\rm EC}\right) = 0. \tag{56}$$

In the ideal equilibrium limit considered here, the actual state realizes the maximum-entropy equilibrium state at the same Landau energy density; hence, the ECEP density equals the equilibrium entropy density $s_{\rm eq}$

$$s_{\rm EC} = s_{\rm eq}. \tag{57}$$

It follows that the comoving entropy is conserved,

$$a^3 s_{\rm eq} = \text{const.} \tag{58}$$

Specifically, writing the equilibrium entropy density as

$$s_{\rm eq}(T) = \frac{2\pi^2}{45} g_{*s}(T) T^3, \tag{59}$$

where $g_{*s}(T)$ denotes the effective entropic degrees of freedom. Conservation of comoving entropy gives

$$a\, g_{*s}^{1/3}(T) T = \text{const.} \tag{60}$$

The temperature scaling deviates from the $T \propto a^{-1}$ relation when $g_{*s}(T)$ varies, but it remains consistent with entropy conservation.

The same result follows directly from standard equilibrium thermodynamics. In the absence of contributions from conserved charges,

$$d\rho = T\, ds, \qquad \rho + p = T s, \tag{61}$$

whereas conservation of the stress–energy tensor in an FLRW universe yields

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0. \tag{62}$$

Combining these relations yields

1) In the Standard Model, both the electroweak transition and the QCD transition at negligible baryon chemical potential are smooth crossovers [52−56].

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$$T\left(\frac{ds}{dt}+3Hs\right)=0, \tag{63}$$

and hence $a^3 s = \text{const}$. The conventional derivation is valid but somewhat abstract. By contrast, the ECEP derivation makes the physical content more transparent. In an equilibrium crossover, the actual isotropic mechanical stress exactly follows the equilibrium-completion target, with no stress mismatch to change the comoving ECEP. Eq. (16) then identifies the conserved ECEP with the thermodynamic entropy of the actual equilibrium state.

This observation also clarifies the entropy increase attributed to the electroweak crossover in Ref. [57]. In that analysis, the Higgs field is taken to follow the temperature-dependent minimum of a finite-temperature effective potential, while the plasma is assumed to thermalize efficiently. Ref. [57] motivates this approximation by assuming that oscillations around the minimum are rapidly damped. In deriving its Eq. (2.22), the field is then taken to track the slowly moving minimum, and its kinetic energy is neglected. Under these assumptions, the evolution is intended to describe an equilibrium crossover to a good approximation. We therefore disagree with the conclusion of Ref. [57] that the crossover itself produces a substantial increase in comoving entropy. The issue can be traced to the thermodynamic treatment of the temperature-dependent effective potential.

We now examine the treatment in Ref. [57] in detail to identify the origin of the discrepancy in the entropy evaluation. Let $f_{\rm eq}(T)$ denote the total equilibrium free-energy density. The total thermodynamic quantities satisfy [58, 59]

$$p_{\rm eq}=-f_{\rm eq}, \tag{64}$$

$$s_{\rm eq}=-\frac{df_{\rm eq}}{dT}=\frac{dp_{\rm eq}}{dT}, \tag{65}$$

$$\rho_{\rm eq}=f_{\rm eq}-T\frac{df_{\rm eq}}{dT}=T\frac{dp_{\rm eq}}{dT}-p_{\rm eq}, \tag{66}$$

and

$$\rho_{\rm eq}+p_{\rm eq}=Ts_{\rm eq}. \tag{67}$$

We apply these relations directly to Eqs. (2.20) and (2.21) of Ref. [57]. Define

$$U(T)\equiv U_\phi\left(\phi_{\rm min}(T)\right), \qquad \rho_R(T)\equiv\frac{\pi^2 g_*}{30}T^4. \tag{68}$$

Neglecting the Higgs kinetic term, as in the derivation of Eq. (2.22) of Ref. [57], Eqs. (2.20) and (2.21) yield the total quantities

$$\rho_{\rm Ref}=U+\rho_R, \qquad \rho_{\rm Ref}+p_{\rm Ref}=\frac{4}{3}\rho_R. \tag{69}$$

They therefore imply the total pressure

$$p_{\rm Ref}=-U+\frac{1}{3}\rho_R. \tag{70}$$

If this pressure characterizes the equilibrium state followed in the tracking approximation, the corresponding total entropy density, energy density, and enthalpy are determined by the equilibrium relations above. Using the same approximation as that employed to derive Eq. (2.22) of Ref. [57], one obtains[1)]

$$s_{\rm eq}=\frac{dp_{\rm Ref}}{dT}=\frac{4\rho_R}{3T}\underline{-\frac{dU}{dT}}, \tag{71}$$

$$\rho_{\rm eq}=T\frac{dp_{\rm Ref}}{dT}-p_{\rm Ref}=U+\rho_R\underline{-T\frac{dU}{dT}}, \tag{72}$$

$$\rho_{\rm eq}+p_{\rm eq}=T\frac{dp_{\rm Ref}}{dT}=\frac{4}{3}\rho_R\underline{-T\frac{dU}{dT}}. \tag{73}$$

Comparison with the expressions used in Ref. [57] is straightforward. Equation (2.20) of that work uses $U+\rho_R$ for the total energy density rather than the expression above, whereas Eq. (2.21) uses $4\rho_R/3$ for the total enthalpy. The underlined terms are therefore precisely the equilibrium contributions that are absent from Eqs. (2.20) and (2.21) of Ref. [57]. Similarly, using $s=(\rho+p)/T$, as in Ref. [57], gives

$$s_{\rm Ref}=\frac{\rho_{\rm Ref}+p_{\rm Ref}}{T}=\frac{4\rho_R}{3T}. \tag{74}$$

Thus, the underlined contribution in $s_{\rm eq}$ is also absent from the entropy density employed there.

The consequences of these omissions can be traced directly through the FLRW energy equation. When the equilibrium terms are retained,

$$0=\dot\rho_{\rm eq}+3H\left(\rho_{\rm eq}+p_{\rm eq}\right). \tag{75}$$

Substituting the above expressions, without combining

1) Ref. [57] uses different values of g_* in different temperature regimes, but the temperature-derivative contribution arising from $g_*(T)$ is not retained in its Eq. (2.22). Threshold-induced modifications to the temperature evolution are therefore explicitly set aside there. Accordingly, within the interval considered, we use the same differential approximation below, $d\rho_R/dT=4\rho_R/T$.

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the underlined terms, yields

$$\begin{aligned} 0 &= \dot{\rho}_R + \frac{dU}{dT}\dot{T} + \frac{d}{dt}\left(\underline{-T\frac{dU}{dT}}\right) + 4H\rho_R \underline{-3HT\frac{dU}{dT}} \\ &= \dot{\rho}_R + \frac{dU}{dT}\dot{T} \underline{-\frac{dU}{dT}\dot{T} - T\frac{d^2U}{dT^2}\dot{T}} + 4H\rho_R \underline{-3HT\frac{dU}{dT}}. \end{aligned} \tag{76}$$

Under the same relativistic approximation, $\rho_R \propto T^4$; hence

$$\dot{\rho}_R = T\frac{d}{dt}\left(\frac{4\rho_R}{3T}\right), \qquad 4H\rho_R = 3HT\left(\frac{4\rho_R}{3T}\right). \tag{77}$$

The FLRW energy equation can therefore be written as

$$\begin{aligned} b0 = T\frac{d}{dt}\left(\frac{4\rho_R}{3T}\right) + \frac{dU}{dT}\dot{T} \underline{-\frac{dU}{dT}\dot{T} - T\frac{d^2U}{dT^2}\dot{T}} \\ +3HT\left(\frac{4\rho_R}{3T}\right) \underline{-3HT\frac{dU}{dT}}. \end{aligned} \tag{78}$$

The term $(dU/dT)\dot{T}$, which arises from differentiating the $U(T)$ term retained in Eq. (2.20) of Ref. [57], is canceled by the first underlined term when the equilibrium energy density is treated consistently. The remaining underlined terms are the time-evolution and expansion terms associated with the equilibrium entropy contribution $-dU/dT$. Using Eq. (71), the equation becomes

$$T\left(\dot{s}_{\rm eq} + 3Hs_{\rm eq}\right) = 0, \tag{79}$$

and hence

$$\frac{d}{dt}\left(a^3 s_{\rm eq}\right) = 0. \tag{80}$$

Ref. [57], however, omits all of the underlined terms above while retaining $U(T)$ itself in the energy density. Omitting the underlined terms from the same equation yields

$$T\frac{d}{dt}\left(\frac{4\rho_R}{3T}\right) + \frac{dU}{dT}\dot{T} + 3HT\left(\frac{4\rho_R}{3T}\right) = 0. \tag{81}$$

With the entropy density $s_{\rm Ref} = 4\rho_R/(3T)$ used in Ref. [57], this becomes

$$\frac{d}{dt}\left(a^3 s_{\rm Ref}\right) = -a^3\frac{1}{T}\frac{dU}{dT}\dot{T}. \tag{82}$$

Thus, omitting the underlined terms required for thermodynamic consistency leads to an apparent increase in comoving entropy. This contradicts the general derivation in Eqs. (61), (62), and (63) and is also not supported, from an alternative perspective, by the ECEP argument.

B. First-Order Phase Transitions

We next consider a first-order cosmological phase transition as a nontrivial benchmark for the application of ECEP. Unlike the equilibrium crossover studied above, the actual state during a first-order phase transition may be strongly out of equilibrium and spatially inhomogeneous on bubble scales. Here, we restrict our analysis to the regime in which these inhomogeneities can nevertheless be coarse-grained into a spatially homogeneous and isotropic FLRW description. This restriction is imposed for the analytic benchmark below; the more strongly inhomogeneous regime discussed in the Introduction would instead require the resolved-hard treatment of Sec. III.D, for which solving the hard region remains an open problem.

More precisely, let $L_{\rm inh}$ denote the largest characteristic scale of the bubble-induced spatial inhomogeneities relevant to the stress-energy distribution. We assume that there exists a coarse-graining scale $\ell_{\rm cg}$ satisfying

$$L_{\rm inh} \ll \ell_{\rm cg} \ll H^{-1}, \tag{83}$$

where H is the Hubble parameter. We may then average over the bubble-induced inhomogeneities using cells of size $\ell_{\rm cg}$ that are small compared with H^{-1}. The resulting effective stress-energy tensor is approximately homogeneous and isotropic on cosmological scales. We further assume that the coarse-graining prescription yields an effective stress-energy tensor $\bar{T}^{\mu\nu}$ that satisfies

$$\nabla_\mu \bar{T}^{\mu\nu} \simeq 0. \tag{84}$$

In the analytic benchmark below, the residual $Q^\nu \equiv \nabla_\mu \bar{T}^{\mu\nu}$ is neglected at the level of accuracy of the coarse-grained description. If this residual were retained, the right-hand side of Eq. (36) would contain the additional term $-u_{L\nu}Q^\nu/T_{\rm EC}$. Hereafter, we treat $\bar{T}^{\mu\nu}$ as the actual stress-energy tensor at the adopted resolution. The analysis is therefore valid only to the extent that the scale hierarchy described above exists and that the coarse-grained FLRW dynamics provides a controlled description to this accuracy.

As an analytic benchmark, we adopt a bag equation of state for the equilibrium thermodynamics [60]. We denote the high- and low-temperature phases by + and −, respectively, and choose the vacuum energy of the low-temperature phase to vanish. Their equilibrium energy densities and pressures are

$$\rho_+(T) = \epsilon + a_+ T^4, \tag{85}$$

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$$p_+(T) = -\epsilon + \frac{a_+ T^4}{3}, \tag{86}$$

and

$$\rho_-(T) = a_- T^4, \tag{87}$$

$$p_-(T) = \frac{a_- T^4}{3}, \tag{88}$$

where $a_+ > a_- > 0$ characterize the relativistic thermal contributions, and $\epsilon > 0$ denotes the false-vacuum energy density. The corresponding entropy densities are

$$s_+(T) = \frac{4}{3} a_+ T^3, \tag{89}$$

$$s_-(T) = \frac{4}{3} a_- T^3. \tag{90}$$

The critical temperature T_c is determined by the equality of the equilibrium pressures,

$$p_+(T_c) = p_-(T_c), \tag{91}$$

which gives

$$\epsilon = \frac{a_+ - a_-}{3} T_c^4. \tag{92}$$

It is convenient to introduce the dimensionless parameters

$$r \equiv \frac{a_-}{a_+}, \qquad q \equiv \frac{1-r}{3}, \tag{93}$$

so that

$$\epsilon = q a_+ T_c^4. \tag{94}$$

At T_c, the two phases coexist at a common pressure

$$p_c \equiv p_+(T_c) = p_-(T_c) = \frac{r}{3} a_+ T_c^4. \tag{95}$$

The limiting energy densities at the two ends of the coexistence interval are

$$\rho_c^- \equiv \rho_-(T_c) = r a_+ T_c^4, \tag{96}$$

and

$$\rho_c^+ \equiv \rho_+(T_c) = (1+q) a_+ T_c^4. \tag{97}$$

Their difference is the latent heat,

$$L \equiv \rho_c^+ - \rho_c^- = 4\epsilon. \tag{98}$$

Thus, the equilibrium first-order phase transition comprises a finite energy-density interval $\rho_c^- \le \rho \le \rho_c^+$ at the fixed temperature T_c and pressure p_c. As discussed in Sec. II.B, in the thermodynamic limit this interval is represented by equilibrium phase coexistence.

For a given coarse-grained Landau energy density ρ, the equilibrium-completion target is determined by the maximum-entropy construction of Sec. II.B. In the present bag model, the corresponding ECEP thermodynamic functions have three branches. For $\rho < \rho_c^-$, the target is the pure low-temperature phase,

$$T_{\rm EC} = \left(\frac{\rho}{a_-}\right)^{1/4}, \qquad p_{\rm EC} = \frac{\rho}{3}, \qquad s_{\rm EC} = \frac{4}{3} a_-^{1/4} \rho^{3/4}. \tag{99}$$

For $\rho_c^- \le \rho \le \rho_c^+$, the entropy-maximizing target is an equilibrium two-phase coexistence state,

$$T_{\rm EC} = T_c, \qquad p_{\rm EC} = p_c, \qquad s_{\rm EC} = \frac{\rho + p_c}{T_c}. \tag{100}$$

If $\lambda_{\rm EC}$ denotes the equilibrium volume fraction of the + phase in this counterfactual coexistence state, it is given by

$$\lambda_{\rm EC} = \frac{\rho - \rho_c^-}{\rho_c^+ - \rho_c^-} = \frac{\rho - \rho_c^-}{4\epsilon}. \tag{101}$$

Finally, for $\rho > \rho_c^+$, the target state is the pure high-temperature phase.

$$T_{\rm EC} = \left(\frac{\rho - \epsilon}{a_+}\right)^{1/4}, \quad p_{\rm EC} = \frac{\rho - 4\epsilon}{3}, \quad s_{\rm EC} = \frac{4}{3} a_+^{1/4} (\rho - \epsilon)^{3/4}. \tag{102}$$

The coexistence fraction $\lambda_{\rm EC}$ pertains only to the counterfactual equilibrium-completion target and should not be identified with the actual phase volume fraction during bubble nucleation and growth. In particular, the actual Universe may still be predominantly, or even entirely, in the supercooled + phase while its coarse-grained energy density already places the ECEP target on the coexistence branch. The three expressions above match continuously at $\rho = \rho_c^{\pm}$, consistent with the regularity of $s_{\rm EC}(\rho)$ established in Sec. II.B.

We next specify the actual coarse-grained evolution

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used in this benchmark. Let $f(t)$ denote the actual volume fraction occupied by the + phase[1)]. We collect all non-vacuum contributions into a radiation-like component with energy density ρ_R and assume that its coarse-grained pressure is $p_R = \rho_R/3$. The total coarse-grained energy density and actual isotropic stress are then

$$\rho = \epsilon f + \rho_R, \tag{103}$$

and

$$p_{\rm act} = -\epsilon f + \frac{\rho_R}{3} = \frac{\rho}{3} - \frac{4\epsilon f}{3}. \tag{104}$$

The conservation of stress-energy in the homogeneous FLRW background gives

$$\frac{d\rho_R}{dt} + 4H\rho_R = -\epsilon\frac{df}{dt}. \tag{105}$$

We denote by t_c and a_c the cosmic time and scale factor associated with the critical temperature T_c, respectively, and define the reduced scale factor A as

$$A \equiv \frac{a}{a_c}, \qquad A(t_c) = 1. \tag{106}$$

With $f(t_c) = 1$ and $\rho_R(t_c) = a_+ T_c^4$, the conservation equation integrates to

$$A^4\rho_R = a_+ T_c^4 + \epsilon M_4(t), \tag{107}$$

in which $M_4(t)$ is defined by

$$M_4(t) \equiv \int_{t_c}^{t} dt' \left[-\frac{df(t')}{dt'}\right] A^4(t'). \tag{108}$$

It is useful to introduce the dimensionless variables

$$\widehat{\rho} \equiv \frac{\rho}{a_+ T_c^4}, \qquad N \equiv \ln A, \tag{109}$$

for which

$$\widehat{\rho} = qf + A^{-4}\left[1 + qM_4(t)\right], \tag{110}$$

and equivalently

$$\frac{d\widehat{\rho}}{dN} = -4\left(\widehat{\rho} - qf\right). \tag{111}$$

The functions $f(t)$ and $a(t)$, and hence the actual trajectory $\widehat{\rho}(t)$, would be determined by the nucleation, bubble-growth, and energy-transfer dynamics, together with the cosmological evolution [2].

For the post-t_c evolution considered here, Eq. (111) determines which of the three equilibrium-completion expressions in Eqs. (99), (100), and (102) is used at a given time. Since $\hat{\rho} - qf = \rho_R/(a_+ T_c^4) > 0$, one has $d\hat{\rho}/dN < 0$ for $\rho_R > 0$. Starting from $\hat{\rho}(t_c) = 1 + q$, the high-energy expression, Eq. (102), is therefore not used at later times. The coexistence expression, Eq. (100), applies while $r \le \hat{\rho} \le 1 + q$. For an asymptotically completed transition, the low-energy expression, Eq. (99), applies once $\hat{\rho} < r$. This classification refers only to the counterfactual equilibrium-completion target; the actual phase composition is described separately by $f(t)$. If the history is extended to the earlier equilibrium era with $T > T_c$, Eq. (102) applies there because $\hat{\rho} = q + (T/T_c)^4 > 1 + q$.

At the adopted coarse-grained resolution, the effective stress-energy tensor is homogeneous and isotropic, so its Landau flow coincides with the FLRW flow, and the shear and anisotropic stress vanish. Eq. (52) therefore applies directly, with all quantities now understood as coarse-grained effective quantities. Since the conservation law in Eq. (84) is imposed only to the accuracy of the effective description, the same qualification applies to the ECEP balance here.

For a fixed comoving volume V_c, define

$$P_{\rm EC} \equiv a^3 V_c s_{\rm EC}. \tag{112}$$

Eq. (52) then gives

$$\frac{dP_{\rm EC}}{dt} = \frac{3Ha^3 V_c}{T_{\rm EC}}\left(p_{\rm EC} - p_{\rm act}\right). \tag{113}$$

Therefore, the evolution of the coarse-grained ECEP is determined directly by the mismatch between the actual isotropic stress in Eq. (104) and the appropriate equilibrium-completion pressure in Eqs. (99), (100), and (102).

We normalize the comoving ECEP to its value at t_c,

$$\Pi_{\rm EC} \equiv \frac{P_{\rm EC}}{P_i}, \qquad P_i \equiv P_{\rm EC}(t_c). \tag{114}$$

Because the initial state at t_c is the equilibrium + phase,

$$P_i = S_i = \frac{4}{3}a_c^3 V_c a_+ T_c^3. \tag{115}$$

Substituting the actual stress from Eq. (104) and the three

1) According to the Johnson-Mehl-Avrami-Kolmogorov (JMAK) equation, $f(t) = e^{-I(t)}$, where $I(t)$ is the fractional extended volume of true-vacuum bubbles at time t. In an FLRW universe, $I(t)$ can be computed once the bubble nucleation rate and wall velocity are known and the scale-factor evolution has been solved [2].

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equilibrium-completion branches into the local ECEP identity yields

$$\frac{d\Pi_{\rm EC}}{dN} = 3qA^3 f\left(\frac{r}{\widehat{\rho}}\right)^{1/4}, \quad \widehat{\rho} < r, \tag{116}$$

$$\frac{d\Pi_{\rm EC}}{dN} = \frac{3A^3}{4}\left(r - \widehat{\rho} + 4qf\right), \quad r \le \widehat{\rho} \le 1+q, \tag{117}$$

$$\frac{d\Pi_{\rm EC}}{dN} = -3qA^3(1-f)(\widehat{\rho}-q)^{-1/4}, \quad \widehat{\rho} > 1+q. \tag{118}$$

After Eq. (111) is used to eliminate f, these three equations can be integrated exactly and matched continuously at the branch boundaries to obtain

$$\Pi_{\rm EC} = A^3 r^{1/4}\widehat{\rho}^{3/4}, \quad \widehat{\rho} < r, \tag{119}$$

$$\Pi_{\rm EC} = \frac{A^3}{4}\left(3\widehat{\rho} + r\right), \quad r \le \widehat{\rho} \le 1+q, \tag{120}$$

$$\Pi_{\rm EC} = A^3(\widehat{\rho}-q)^{3/4}, \quad \widehat{\rho} > 1+q. \tag{121}$$

For an idealized, asymptotically completed transition, we take

$$f \to 0, A^4 f \to 0, M_4(t) \to M_4 < \infty. \tag{122}$$

Eq. (110) then gives

$$A^4\widehat{\rho} \longrightarrow 1 + qM_4, \widehat{\rho} \longrightarrow 0. \tag{123}$$

The equilibrium-completion target therefore approaches the pure $-$ branch. From Eq. (119),

$$\frac{P_{{\rm EC},f}}{P_i} = r^{1/4}(1+qM_4)^{3/4}, \tag{124}$$

where $P_{{\rm EC},f}$ denotes the asymptotic comoving ECEP. We further assume that, after completion and relaxation, the actual system asymptotically realizes the $-$ phase as a stable equilibrium state. Denoting the comoving thermodynamic entropy of this idealized final equilibrium state by S_f, equilibrium consistency implies $P_{{\rm EC},f} = S_f$. Together with $P_i = S_i$, this yields

$$\frac{S_f}{S_i} = r^{1/4}\left(1 + \frac{1-r}{3}M_4\right)^{3/4}. \tag{125}$$

Here, the dependence on the phase-conversion history is encapsulated in

$$M_4 = \int_{t_c}^{\infty} dt \left(-\frac{df}{dt}\right) A^4(t). \tag{126}$$

The A^4 weighting makes M_4 especially sensitive to phase conversion occurring at late times. Thus, within this benchmark, the final entropy increase depends on the detailed history of the phase transition only through this single weighted moment.

The second law of thermodynamics requires $S_f \ge S_i$; therefore, we can derive a constraint on M_4 from Eq. (125):

$$M_4 \ge M_{4,\min}(r) \equiv \frac{3\left(r^{-1/3}-1\right)}{1-r} > 1. \tag{127}$$

This, in turn, imposes a constraint on the allowed history of the phase conversion $f(t)$ and the scale factor evolution $A(t)$. Equality is reached for reversible phase coexistence at T_c, with $f(A) = (A^{-3}-r)/(1-r)$ for $1 \le A \le r^{-\frac{1}{3}}$ and $f = 0$ for $A > r^{-\frac{1}{3}}$. This corresponds to $S_f = S_i$, which saturates the constraint. Ultimately, $f(t)$ should be computed by solving the phase transition dynamics with inputs such as the nucleation rate, with the above constraint serving as a necessary condition for physically realizable phase transition dynamics.

When the phase transition dynamics can be solved, the same final entropy ratio can also be obtained from the conventional FLRW energy evolution with equilibrium thermodynamics[1]. The role of ECEP here is primarily conceptual and diagnostic: even if the intermediate nonequilibrium state does not have a well-defined entropy, the evolution of its entropy capacity can still be analyzed consistently through the phase-coexistence target stage associated with a first-order phase transition.

V. Discussion and Conclusions

In this work, we introduced ECEP to characterize the thermalization capacity of nonequilibrium states in a generic curved spacetime. This concept is motivated by entropy production in a strongly supercooled first-order phase transition in the early universe, which, when pushed toward the boundary of a successful transition, can lead to significant inhomogeneities due to large bubbles[2] and thereby invalidate the conventional use of a

1) Related conventional treatments of entropy release and relic dilution in cosmological first-order phase transitions can be found in [21, 61, 62].

2) Large bubbles can induce significant general-relativistic and large-scale inhomogeneity effects [14, 16, 38, 39].

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homogeneous FLRW cosmology. The beginning and endpoint of such a strongly supercooled phase transition are relatively simple thermal states that admit a homogeneous FLRW background description, whereas the intermediate stage can be substantially more complex. The lack of a well-defined global notion of energy in curved spacetime [40] obstructs a simple evaluation of entropy increase in such a case. Nevertheless, for these complex intermediate states, one may first characterize the thermalization capacity locally via the energy density in the Landau frame and then integrate over a finite domain to characterize the entropy capacity for a specified spacelike region. The essence of ECEP lies in the choice of the equilibrium-completion target using the maximum-entropy principle. This makes ECEP non-teleological and regular across the phase-coexistence stage of a first-order phase transition.

The central result of this work is the exact local ECEP identity, Eq. (36), which is valid in a generic curved spacetime and can be derived using simple thermodynamic relations and the definition of the Landau frame. The identity also has a clear physical interpretation in terms of the response mismatch between the actual and target stresses. It can be further integrated to obtain a finite-domain balance relation. If, on the initial and final spacelike hypersurfaces, the ECEP coincides with the actual entropy density, as in the case of Eq. (16), this formalism can in principle be employed to study entropy production in nonequilibrium processes in curved spacetime. Although the notion of entropy is not defined in the intermediate stage, ECEP can help elucidate how the entropy capacity is created and realized continuously during the process.[1)]

We have applied the concept of ECEP to two canonical examples of cosmological phase transitions. The first is a smooth crossover transition. Typical examples include the electroweak phase transition and the QCD phase transition in the Standard Model. We use the ECEP concept to show explicitly how the comoving entropy and its capacity remain conserved owing to the absence of stress mismatches, and thus we disagree with certain claims in the existing literature that significant entropy is produced during the electroweak crossover. In this example, ECEP provides more physical intuition than the abstract conventional thermodynamic derivation. The second example is a more complex first-order phase transition, which features a phase-coexistence stage as the equilibrium-completion target. An explicit calculation can be carried out if we assume that the inhomogeneities caused by the phase transition are small, so that we may still work in a homogeneous FLRW background and use the coarse-grained effective stress-energy tensor. The entropy production factor $\frac{S_f}{S_i}$ can be obtained by integrating the local ECEP identity in such a case, demonstrating the self-consistency of the ECEP concept.

How to compute ECEP in the hard region remains an open problem. Here, the hard region denotes regimes in which simple closure relations for the components of the stress-energy tensor are absent. It is conceivable that one may need to consider typical nonequilibrium processes case by case or rely on numerical methods. However, because ECEP is formulated locally from the outset, when the coarse-graining scale is too small, ECEP may fail to capture the entropy capacity for thermalization events that occur much later, when two streams of matter meet. This calls for an improvement or revision of ECEP that accounts for such cases, which we leave for future work.

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A. ECEP, Thermalization Connectivity and Conserved Charges

Here, we consider two possible complications that are not included in the ECEP construction discussed in the main text but may arise in certain theories: thermalization connectivity and conserved charges.

The thermalization connectivity relevant to ECEP is defined pointwise. The actual local state at a spacetime point x provides the initial data for an auxiliary counterfactual problem. Within the same underlying theory, this state is allowed to relax in an auxiliary Minkowski setting over a prescribed equilibrium-completion timescale τ_{EC}.

A set of retained degrees of freedom is said to form a thermalization-connected sector S at x if, in this auxiliary problem, they can establish a common thermal equilibrium within τ_{EC}. Degrees of freedom that cannot establish a common thermal state on this timescale are treated as belonging to different sectors[2)]. The resulting sector partition can depend on both the local state and the prescribed τ_{EC}. In the following, energy-momentum exchange between distinct sectors is neglected at the adopted level of approximation. Within this approximation, the stress-energy tensor $T_S^{\mu\nu}$ of each sector obeys

1) As discussed in Sec. II.B, $s_{\mathrm{EC}}(\rho)$ is continuous and differentiable even when first-order phase transitions are allowed, so its evolution is continuously characterized by Eq. (36). This is why we say "the entropy capacity is created and realized continuously". For a finite domain, the sign of the endpoint change is determined by the full ECEP balance and may also depend on the boundary flux.

2) A hidden sector that interacts very weakly with the Standard Model [63−69] is a canonical example.

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$$\nabla_\mu T_S^{\mu\nu} = 0. \tag{A.1}$$

After the thermalization-connected sectors have been identified, the ECEP construction is performed separately for each sector. Assuming that the stress-energy tensor in sector S has a unique future-directed timelike Landau eigenvector, we define

$$T_S^{\mu\nu} u_{S\nu} = -\rho_{L,S} u_S^\mu, \qquad u_S^\mu u_{S\mu} = -1, \tag{A.2}$$

and introduce

$$h_S^{\mu\nu} = g^{\mu\nu} + u_S^\mu u_S^\nu, \qquad p_{\mathrm{act},S} = \frac{1}{3} h_{S\mu\nu} T_S^{\mu\nu}, \tag{A.3}$$

$$\pi_S^{\mu\nu} = h_{S\,\alpha}^{\mu} h_{S\beta}^{\nu} T_S^{\alpha\beta} - p_{\mathrm{act},S} h_S^{\mu\nu}. \tag{A.4}$$

Here, $\rho_{L,S}$ is the Landau-frame energy density of the sector, whereas $p_{\mathrm{act},S}$ and $\pi_S^{\mu\nu}$ are, respectively, the isotropic and traceless anisotropic components of its actual spatial stress.

Conserved charges[1)] may impose additional constraints. A precise characterization also requires consideration of a pointwise auxiliary problem. For a given sector, an additive global charge is retained as an active conserved charge if it is either strictly conserved or changes by a negligible amount during the counterfactual relaxation over τ_{EC}. If the corresponding charge-changing processes act sufficiently strongly on this timescale, that charge is not retained as an independent constraint on the equilibrium-completion target.

The timescale τ_{EC} also determines whether a metastable phase should be left during the auxiliary completion. A sector may reach a common temperature within τ_{EC}, whereas the phase-conversion timescale may be much longer. In this case, the metastable phase is kept as an effective constraint in the completion problem. If phase conversion can occur within τ_{EC}, the stable phase is included in the admissible set. The entropy maximum is then evaluated over the states allowed by these constraints.

Let A label the independent conserved charges of sector S, and let $J_{A,S}^\mu$ denote their currents, which obey

$$\nabla_\mu J_{A,S}^\mu = 0. \tag{A.5}$$

In the Landau frame of the sector, we write

$$J_{A,S}^\mu = n_{A,S} u_S^\mu + j_{A,S}^\mu, \tag{A.6}$$

with

$$n_{A,S} = -u_{S\mu} J_{A,S}^\mu, \qquad u_{S\mu} j_{A,S}^\mu = 0. \tag{A.7}$$

The spatial current $j_{A,S}^\mu$ does not necessarily vanish and represents charge transport with respect to the Landau energy flow.

The definition of the equilibrium-completion target is based on the partition into sectors determined by thermalization connectivity and the active conserved charges. In each sector, this target is the maximum-entropy equilibrium state with the same instantaneous Landau energy density $\rho_{L,S}$ and active charge densities $\{n_{A,S}\}$ as the actual local state. The corresponding ECEP entropy density is therefore

$$s_{\mathrm{EC},S} = s_{\mathrm{EC},S}\left(\rho_{L,S}, \{n_{A,S}\}\right). \tag{A.8}$$

The corresponding temperature $T_{\mathrm{EC},S}$, pressure $p_{\mathrm{EC},S}$, and chemical potentials $\mu_{A,S}^{\mathrm{EC}}$ satisfy

$$d\rho_{L,S} = T_{\mathrm{EC},S}\, ds_{\mathrm{EC},S} + \sum_A \mu_{A,S}^{\mathrm{EC}}\, dn_{A,S}, \tag{A.9}$$

and

$$\rho_{L,S} + p_{\mathrm{EC},S} = T_{\mathrm{EC},S}\, s_{\mathrm{EC},S} + \sum_A \mu_{A,S}^{\mathrm{EC}} n_{A,S}. \tag{A.10}$$

The partition into thermalization-connected sectors, together with the active conserved charges and other retained completion constraints for each sector, specifies the equilibrium-completion structure associated with the given local state. Consider a spacetime region in which this structure remains fixed and the relevant equilibrium thermodynamic branches are differentiable. We define the expansion scalar and shear tensor of the sector's Landau congruence by

$$\theta_S \equiv \nabla_\mu u_S^\mu, \qquad \sigma_S^{\mu\nu} \equiv h_{S\,\alpha}^{\mu} h_{S\beta}^{\nu} \nabla^{(\alpha} u_S^{\beta)} - \frac{1}{3}\theta_S h_S^{\mu\nu}. \tag{A.11}$$

The conservation of the stress-energy tensor gives

$$u_S^\mu \nabla_\mu \rho_{L,S} + \left(\rho_{L,S} + p_{\mathrm{act},S}\right)\theta_S + \pi_S^{\mu\nu}\sigma_{S\mu\nu} = 0, \tag{A.12}$$

whereas each active conserved charge satisfies

$$u_S^\mu \nabla_\mu n_{A,S} + n_{A,S}\theta_S + \nabla_\mu j_{A,S}^\mu = 0. \tag{A.13}$$

1) We consider only global $U(1)$ charges in this work. Extensions to non-Abelian or gauge charges are more complicated and are therefore not addressed here.

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The ECEP current for sector S can still be introduced

$$J^{\mu}_{\mathrm{EC},S} \equiv s_{\mathrm{EC},S}\, u^{\mu}_{S}. \tag{A.14}$$

Combining the thermodynamic relations with the energy and charge equations above yields the generalized local ECEP identity.

$$T_{\mathrm{EC},S}\nabla_{\mu}J^{\mu}_{\mathrm{EC},S} = \left(p_{\mathrm{EC},S} - p_{\mathrm{act},S}\right)\theta_{S} - \pi^{\mu\nu}_{S}\sigma_{S\mu\nu} + \sum_{A}\mu^{\mathrm{EC}}_{A,S}\nabla_{\mu}j^{\mu}_{A,S}. \tag{A.15}$$

The first two terms on the right-hand side correspond to the isotropic-stress mismatch and anisotropic stress–shear terms discussed in Sec. III.B, here restricted to a single thermalization-connected sector. The additional term describes the redistribution of active conserved charges relative to the Landau energy flow. As in the main text, Eq. (142) is exact in an algebraic sense: once the sector partition and active conserved charges have been fixed, it follows from equilibrium thermodynamics and local energy and charge conservation.

For multiple thermalization-connected sectors, the ECEP remains defined separately for each sector. For a finite domain D on a spacelike hypersurface Σ, the ECEP associated with sector S is

$$P_{\mathrm{EC},S}[D,\Sigma] = -\int_{D} J^{\mu}_{\mathrm{EC},S}\, n_{\mu}\, d\Sigma. \tag{A.16}$$

The worldtube balance in Sec. III.C then applies separatelytoeachsector,withitsownECEPcurrentandlocalidentity.

Finally, the local differential identity presented above assumes that the partition into sectors and the set of active conserved charges remain fixed. If either changes as the actual local state evolves, the corresponding ECEP branches should be evaluated separately on both sides, and any discontinuous contribution must be included separately in a finite-domain balance. For an isolated change on a hypersurface Σ_k, we define the corresponding jump by

$$\Delta P_{\mathrm{EC},k} \equiv P^{+}_{\mathrm{EC}}[D_k,\Sigma_k] - P^{-}_{\mathrm{EC}}[D_k,\Sigma_k], \tag{A.17}$$

where the two values are evaluated using the completion structures on either side of Σ_k. The worldtube balance is applied separately between successive changes, and the jumps $\Delta P_{\mathrm{EC},k}$ are then added to the integrated balance.

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